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Asset Pricing Model with Short-Sale Restrictions: The Case of Asian Property Markets

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Asset pricing models have been used extensively in the recent real estate literature to evaluate real estate performance and estimate required rates of return of properties. In this paper, we show that the CAPM and its variants will derive a biased result when short sales are not allowed in the market. This problem is particularly serious for Asian property markets where investors are not able to short sell real estate indexes as a substitute for short selling real properties. We also demonstrate that the bias resulting from the short-sale constraint is related to the supply-and-demand conditions in the local market.

Keywords

Asian property markets, asset pricing model, short-sale restriction.

Introduction

It is well known that in a perfect capital market, the traditional Capital Asset Pricing Model (CAPM) derived by Sharpe (1964) requires the use of several highly stylized assumptions. Among the assumptions subject to most objections is that investors are able to take a short position of any size in any asset. Clearly, this assumption does not hold in the real world. This problem (no short-sale restriction) does not go away with the development of more recent asset pricing models. For example, the 3-factor model advocated by Fama and French (1992) also uses beta as one of the factors in their estimation equation. In other words, this unrealistic assumption applies to any asset-pricing model that implies a linear relationship between expected return and beta.

Following Sharpe (1964), several researchers have studied the effect of the short-sale restriction on the predictions of the CAPM. For example, Lintner (1971) analyses the effect of short selling and margin requirements on the CAPM and proves that a restriction on the use of the short sale proceeds will not affect the optimal demand from investors. However, he also speculates that "when short selling is prohibited for any investors in the market, the market equilibrium set of current prices will not be the same as when there were no restrictions on short selling." (p. 1193). Ross (1977), using a numerical example, demonstrated that the traditional CAPM breaks down if there are short-sale restrictions in the market. Finally, Dybvig (1984) demonstrated that the mean-variance efficient-frontier could be kinked if short sales are constrained in the financial markets. Recently, Choie and Hwang (1994) reported that stocks with a large short position consistently underperform the market, implying that the prices of those stocks are higher than that predicted by the CAPM. Apparently, both theoretical and empirical evidence indicate that the assumption of no restrictions on short sales could be a problem when applying the CAPM in the real world.

However, even with a clear understanding of the limitation, the CAPM and its variants are still being used extensively in the finance literature. This is probably because finance researchers feel that the problem is not serious enough for them to discard the model. Indeed, while it is true that investors will not be able to take a large short position in a particular stock, short sales of reasonable sizes are still allowed for most stocks in the market. Given this, it might make sense for the empirical finance literature to ignore the assumption of unlimited short sales when analyzing asset returns.

The problems appeared when the CAPM and its variants were used to study real estate returns in the early 1980s. We know for a fact that given the current institutional environment, investors cannot short sell real properties. Given this clear violation of the assumption behind the CAPM, it might not be appropriate for researchers to apply the model directly to real estate markets without modifications. However, it can be argued that although investors cannot short sell real properties directly, they can short sell REIT (Real Estate Investment Trust) stocks. Since REITs hold real properties, investors should be able to construct portfolios of REIT stocks that mimic the movements of particular real estate markets. Because of the ability to short sell REIT stocks, the use of the CAPM (and its variants) to evaluate property performance may still be an acceptable approach.

However, the real problem began when researchers started to apply the CAPM to Asian real estate markets in the early 1990s. Clearly, at the current time, investors cannot short sell real properties in these markets. In addition, there are not enough REITs for investors to construct indexes that will mimic the movements of the property markets. Given this, special caution should be taken when applying the CAPM and its variants to study property markets in Asia.

Indeed, researchers have attempted to revise the CAPM to meet the special characteristics of real estate markets. For example, Liu, Grissom, and Hartzell (1990) investigated the consequences of several imperfections (liquidity and consumption) associated with real estate markets on pricing from a CAPM context. Lai, Wang, Chan, and Lee (1992) also provided a simple method to construct an optimal portfolio with short-sale restrictions on real estate assets. This paper demonstrates explicitly the impact of short-sale restrictions on the pricing of real estate assets.

The following section discusses the general framework of our model. The impacts of short-sale restrictions are demonstrated using a simple k+2 asset-pricing model in the third section. The fourth section discusses the implications of the model based on the condition of local markets. The last section concludes.

Model Framework

In our model, the opportunity set of an investor consists of *n* risky financial assets, a risk-free financial asset, and *m* risky real estate assets. The price change for the *i*-th financial asset is specified as:

$$\frac{dP_i^a}{P_i^a} = R_i^a(\mathbf{S},t)\,dt + \sigma_i^a(\mathbf{S},t)\,d\,z_i^a,\ \forall\, i = 1, 2, ..., n, \tag{1}$$

while the price change for the *i*-th real estate asset is specified as:

$$\frac{dP_i^r}{P_i^r} = R_i^r(\mathbf{S},t)\,dt + \sigma_i^r(\mathbf{S},t)\,dz_i^r,\ \forall\, i = 1, 2, ..., m, \tag{2}$$

where P_i^a and P_i^r are the prices of the *i*-th financial asset and the *i*-th real estate asset, respectively. $R_i^a(\mathbf{S},t)$ and $R_i^r(\mathbf{S},t)$ are the instantaneous expected rates of return on the *i*-th financial asset and real estate asset, respectively. $\mathbf{S}$ is a vector of state variables. $\sigma_i^a(\mathbf{S},t)$ and $\sigma_i^r(\mathbf{S},t)$ are the instantaneous standard deviations of rates of return on the *i*-th financial asset and real estate asset, respectively. dz_i^a and dz_i^r are the standard Gauss Wiener processes. The expected return of the risk free financial asset is specified as $R_0^a(\mathbf{S},t) = r(\mathbf{S},t)$, with $\sigma_0^a(\mathbf{S},t) = 0$. The change of the state variables vector $\mathbf{S}$ is assumed to follow vector Ito process, or:

$$\mathbf{dS} = \mathbf{F}(\mathbf{S},\mathbf{t})dt + \mathbf{G}(\mathbf{S},\mathbf{t})\,d\mathbf{Z}_s\,, \tag{3}$$

where $\mathbf{F}(\mathbf{S},\mathbf{t})$ is a *k*x1 vector of the instantaneous expected change in the state variables, $\mathbf{G}(\mathbf{S},\mathbf{t})$ is a *k*x*k* diagonal matrix of instantaneous standard deviation, and $d\mathbf{Z}_s$ is a *k*x1 vector of standard Gauss Wiener process.

In this economy, an investor's problem is to allocate her/his wealth among the investment opportunities (which includes the *n* risky financial assets, a risk-free financial asset, and *m* risky real estate assets), subject to the budget constraint, or:

$$W_0 = \mathbf{X}_a'\,\mathbf{P}_a^0 + \mathbf{X}_r'\,\mathbf{P}_\mathrm{r}^0 + X_0 + C\,, \tag{4}$$

where $\mathbf{P}_a^0$ and $\mathbf{P}_r^0$ are the *n*x1 and *m*x1 vectors of financial and real estate asset prices at time 0, respectively. $\mathbf{X}_a'$ and $\mathbf{X}_r'$ (the transpose of $\mathbf{X}_a$ and $\mathbf{X}_r$) are 1x*n* and 1x*m* vectors of demand for the financial and real estate assets, respectively. The *i*-th elements of $\mathbf{X}_a$ and $\mathbf{X}_r$ are X_{a_i} and X_{r_i}, respectively. X_0 is the demand for the risk-free asset. To simplify the presentation, the current price of the risk free asset is normalized to be one. *C* is the initial consumption.

The wealth change of an investor depends upon her/his consumption rate *c*, rate of wage income *Y*, and portfolio decisions $\mathbf{X}_a$, $\mathbf{X}_r$ and X_0. We assume that the consumption rate c and rate of wage income Y are non-stochastic. Thus, the wealth change of an investor is determined by:

$$dW = \mathbf{X}'_a\, d\mathbf{P}_a + \mathbf{X}'_r\, d\mathbf{P}_r + r\, X_0 dt + (Y - \mathrm{c})dt, \tag{5}$$

where $d\mathbf{P}_a$ and $d\mathbf{P}_r$ are nx1 and mx1 vectors with the i-th element dP_i^a and dP_i^r, respectively. r is the rate of return on the risk-free asset. Given this, an investor's optimization problem can be formulated as to maximize the Neumann-Morgenstein utility of consumption, or to maximize:

$$E_0\left[\int_0^{\mathrm{T}} U(c,t)dt + B(W(T), \mathbf{S}(T), T)\right], \tag{6}$$

where $U(c, t)$ is the utility function of the investor at time t and is a function of the consumption rate c, B(·) is the bequest function of the investor at the time of death T, $W(T)$ is the wealth of the investor at time T, and $\mathbf{S}(T)$ is a vector of state variables at time T. The utility function $U(\cdot)$ and bequest function $B(\cdot)$ are assumed to be time-additive, strictly concave, and differentiable.

Given the above equation, the optimal indirect utility function $J(W, \mathbf{S}, t)$ for the investor can be defined as:

$$J(W, \mathbf{S}, t) = Max\ E_t \int_{\mathrm{t}}^{\mathrm{T}} U(c,t)dt + B(W(T), \mathbf{S}(T), T)), \tag{7}$$

subject to the wealth and state variable changes specified in Equations (5) and (3), respectively. Merton (1971, 1973) demonstrated that the optimal decisions of consumption and investment under the constraint imposed by Equations (3) and (5) must satisfy the Bellman equation, or:

$$\begin{aligned} 0 = Max\ \{&U(c,t) + J_W\ \{[\mathbf{X}_a^{P'}(\mathbf{R}_a - \mathbf{r}) + \mathbf{X}_r^{P'}(\mathbf{R}_r - \mathbf{r}) + \mathrm{r}(W_0 - C)] + Y - c\} \\ &+ J_t + J_S\mathbf{F} + \frac{1}{2}[J_{WW}(\mathbf{X}_a^{P'}\boldsymbol{\Omega}_{R_aR_a}\mathbf{X}_a^P + 2\,\mathbf{X}_a^{P'}\boldsymbol{\Omega}_{R_aR_r}\mathbf{X}_r^P + \mathbf{X}_r^{P'}\boldsymbol{\Omega}_{R_rR_r}\mathbf{X}_r^P)] \\ &+ \mathbf{J}_{WS}[\boldsymbol{\Omega}_{R_aS}\mathbf{X}_a^P + \boldsymbol{\Omega}_{R_rS}\mathbf{X}_r^P] + \frac{1}{2}[\Sigma \mathrm{J}_{S_iS_j} g_{ij}]\}, \end{aligned} \tag{8}$$

where $\mathbf{R}_a\mathbf{-r}$ and $\mathbf{R}_r\mathbf{-r}$ are the nx1 and mx1 vectors of the expected excess returns on the financial and real estate asset, respectively. $\mathbf{X}_a^{P'}$ and $\mathbf{X}_r^{P'}$ (the transpose of $\mathbf{X}_a^P$ and $\mathbf{X}_r^P$) are the 1xn and 1xm vectors with the i-th elements $X_{a_i}\mathrm{P}_{a_i}^0$ and $X_{r_i}\mathrm{P}_{r_i}^0$, respectively. $\boldsymbol{\Omega}_{R_aR_a}$ and $\boldsymbol{\Omega}_{R_rR_r}$ are the nxn and mxm covariance matrixes of returns on financial and real estate assets, respectively. $\mathbf{\Omega}_{R_aR_r}$ is the nxm covariance matrix of returns on financial and real estate assets. $\mathbf{\Omega}_{R_aS}$ and $\mathbf{\Omega}_{R_rS}$ are the kxn and kxm covariance matrixes of financial and real estate asset returns and the state variables, respectively. $g_{ij}dt$ is the instantaneous covariance between the state variable changes ds_i and ds_j. The J functions with subscripts are the partial derivatives.

A Simple k+2 Asset Pricing Model

In this paper, we allow investors to short sell financial assets, but not real estate assets. Given this constraint, the demand of a real estate asset X_{r_i} must be bounded between zero and the available supply in the market q_{r_i}. That is,

$$X_{r_i}(X_{r_i} - q_{r_i}) \leq 0, \forall i = 1, 2, ..., m. \tag{9}$$

In other words, since there is no short sale, the demand for real estate assets must be non-negative and cannot be more than the available supply in the market. However, the constraints imposed by Equation (9) do not apply to financial assets since short sales are allowed. Allowing short sales for financial assets implies that investors can take a short (or long) position when the price of a financial asset is too high (or too low), thus ensuring that the asset price will not deviate too much from the equilibrium price. However, since short sales are not allowed for real estate assets, this condition does not hold in the real estate market.

Given the constraints imposed by Equation (9), the three necessary conditions for the maximization of Equation (8) are:

$$U_c = J_W, \tag{10}$$

$$0 = J_W \begin{bmatrix} \overline{\mathbf{P}}_a - (1+r)\,\mathbf{P}_\mathbf{a}^\mathbf{0} \\ \overline{\mathbf{P}}_r - (1+r)\,\mathbf{P}_\mathbf{r}^\mathbf{0} \end{bmatrix} + J_{WW} \begin{bmatrix} \mathbf{\Omega}_{P_aP_a} & \mathbf{\Omega}_{P_aP_r} \\ \mathbf{\Omega}_{P_rP_a} & \mathbf{\Omega}_{P_rP_r} \end{bmatrix} \begin{bmatrix} \mathbf{X}_a^* \\ \mathbf{X}_r^* \end{bmatrix} + J_{WS} \begin{bmatrix} \mathbf{\Omega}_{P_aS} \\ \mathbf{\Omega}_{P_rS} \end{bmatrix} + \begin{bmatrix} 0 \\ -\mathbf{B}(2\mathbf{X}_r^* - \mathbf{Q}_r) \end{bmatrix}, \tag{11}$$

and

$$b_i \mathrm{X}_{r_i}^* (\mathrm{X}_{r_i}^* - q_{r_i}) = 0, \forall \mathrm{i} = 1, 2, ..., m, \tag{12}$$

where $\overline{\mathbf{P}}_a$ and $\overline{\mathbf{P}}_r$ are nx1 and mx1 vectors of the expected financial and real estate asset prices, respectively. $\boldsymbol{\Omega}_{P_aP_a}$ and $\boldsymbol{\Omega}_{P_rP_r}$ are the nxn and mxm covariance matrixes of financial asset and real estate asset prices, respectively. $\boldsymbol{\Omega}_{P_aP_r}$ is the nxm covariance matrix of financial asset prices and real estate asset prices. $\boldsymbol{\Omega}_{P_aS}$ and $\boldsymbol{\Omega}_{P_rS}$ are kxn and kxm covariance matrixes of financial asset prices and the state variables, and real estate asset prices and the state variables, respectively. $\boldsymbol{B}$ is an mxm diagonal Lagrangian multiplier matrix with b_i as the i-th diagonal element of the matrix, where b_i must be $\geq$ 0. $\mathbf{X}_a^*$ and $\mathbf{X}_r^*$ are the optimal demand vectors for financial and real estate assets, respectively.

Equation (12) is the Kuhn-Tucker condition. Given this, the i-th element $-b_i\,(2X_{r_i}^* - q_{r_i})$ in the vector of $-\mathbf{B}\,(2\mathbf{X}_r^* - \mathbf{q}_r)$ in Equation (11) must be:

$$b_i = 0 = -b_i\,(2X_{r_i}^* - q_{r_i}),\ \text{if } X_{r_i}^* \text{ is not binding,} \tag{13}$$

$$-b_i\,(2X_{r_i}^* - q_{r_i}) = -b_i\,q_{r_i} < 0,\ \text{if } X_{r_i}^* \text{ is binding at } q_{r_i},\ \text{or} \tag{14}$$

$$-b_i\,(2X_{r_i}^* - q_{r_i}) = b_i\,q_{r_i} > 0,\ \text{if } X_{r_i}^* \text{ is binding at } 0, \tag{15}$$

where $X_{r_i}^*$ is the optimal demand for the real estate asset i and q_{r_i} is the available supply in the market of the real estate asset i. Equation (13) indicates that if the optimal allocation of real estate is not binding at 0 or q_{r_i}, then the Lagrangian multiplier b_i must be zero. Equation (14) denotes that the optimal demand for a real estate asset is binding at $\mathrm{X}_{r_i}^* = q_{r_i}$. In other words, the optimal demand for a real property will be greater than q_{r_i} if the supply is not limited at the q_{r_i} level. Similarly, Equation (15) shows that the optimal demand for a real property is binding at 0. Under this condition, investors will short sell the real property (i.e. negative demand) if short selling is allowed. However, because short selling is not allowed, the optimal demand is restricted to 0 (and hence is binding at 0). Simply put, Equations (14) and (15) specify the conditions under which the optimal demand $X_{r_i}^*$ is binding at q_{r_i} and 0, respectively.

Dividing Equation (11) by J_{WW} and aggregating the demand of all investors yields:

$$\begin{bmatrix}\Sigma\,\mathbf{X}_a^*\\ \Sigma\,\mathbf{X}_r^*\end{bmatrix} = \begin{bmatrix}\mathbf{\Omega}_{P_aP_a} & \mathbf{\Omega}_{P_aP_r}\\ \mathbf{\Omega}_{P_rP_a} & \mathbf{\Omega}_{P_rP_r}\end{bmatrix}^{-1}(\Sigma\frac{-J_W}{J_{WW}})\begin{bmatrix}\overline{\mathbf{P}}_a - (1+\mathrm{r})\,\mathbf{P}_{\mathbf{a}}^{\mathbf{0}}\\ \overline{\mathbf{P}}_r - (1+\mathrm{r})\,\mathbf{P}_{\mathbf{r}}^{\mathbf{0}}\end{bmatrix} \tag{16}$$
$$+\ (\Sigma\frac{-J_{WS}}{J_{WW}})\begin{bmatrix}\mathbf{\Omega}_{P_aS}\\ \mathbf{\Omega}_{P_rS}\end{bmatrix} + \begin{bmatrix}0\\ \Sigma\,[\mathbf{B}(2\mathbf{X}_r^* - \mathbf{q}_r)\,/\,J_{WW}\,]\end{bmatrix},$$

where $\Sigma\,\mathbf{X}_a^*$ is the total demand for financial assets and $\Sigma\,\mathbf{X}_r^*$ is the total demand for real properties. An asset pricing model with a short sale restriction on real properties can be obtained by multiplying the covariance matrix of financial asset and real estate asset prices to both sides of Equation (16), and then dividing the result by $\Sigma\frac{-J_W}{J_{WW}}$. With a simple manipulation, we have:

$$\begin{bmatrix}\overline{\mathbf{P}}_a - (1+r)\mathbf{P}_{\mathbf{a}}^0\\ \overline{\mathbf{P}}_r - (1+r)\mathbf{P}_{\mathbf{r}}^0\end{bmatrix} = \frac{1}{\Sigma\frac{-J_W}{J_{WW}}}\begin{bmatrix}\mathbf{\Omega}_{P_aP_a} & \mathbf{\Omega}_{P_aP_r}\\ \mathbf{\Omega}_{P_rP_a} & \mathbf{\Omega}_{P_rP_r}\end{bmatrix}\begin{bmatrix}\Sigma\mathbf{X}_a^*\\ \Sigma\mathbf{X}_r^*\end{bmatrix} + \frac{\Sigma\frac{J_{WS}}{J_{WW}}}{\Sigma\frac{-J_W}{J_{WW}}}\begin{bmatrix}\mathbf{\Omega}_{P_aS}\\ \mathbf{\Omega}_{P_rS}\end{bmatrix} \tag{17}$$
$$+\frac{-1}{(\Sigma\frac{-J_W}{J_{WW}})}\begin{bmatrix}\mathbf{0}\\ \Sigma[\mathbf{B}(2\mathbf{X}_r^* - \mathbf{q}_r)/J_{WW}]\end{bmatrix}.$$

Equation (17) indicates that the asset-pricing model for financial assets is:

$$\mathbf{P}_{\mathbf{a}}^{\mathbf{0}} = \frac{\overline{\mathbf{P}}_a - \lambda[\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{a}},\widetilde{V}_{\mathrm{a}}) + \mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{a}},\widetilde{V}_r)] - \boldsymbol{\delta}\mathbf{Cov}\ (\widetilde{\mathbf{P}}_{\mathbf{a}},\widetilde{\mathbf{S}})}{1+r} \tag{18}$$
$$= \frac{\overline{\mathbf{P}}_a - \lambda_{\mathrm{a}}\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{a}},\widetilde{R}_m^a) - \lambda_{\mathrm{r}}\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{a}},\widetilde{R}_m^r) - \boldsymbol{\delta}\mathbf{Cov}\ (\widetilde{\mathbf{P}}_{\mathbf{a}},\widetilde{\mathbf{S}})}{1+r},$$

while the asset pricing model for real estate assets is:

$$\mathbf{P}_{\mathbf{r}}^{\mathbf{0}} = \frac{\overline{\mathbf{P}}_r - \lambda[\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{r}},\widetilde{V}_{\mathrm{a}}) + \mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{r}},\widetilde{V}_{\mathbf{r}})] - \boldsymbol{\delta}\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{r}},\widetilde{\mathbf{S}}) - \boldsymbol{\pi}}{1+r} \tag{19}$$
$$= \frac{\overline{\mathbf{P}}_r - \lambda_{\mathrm{a}}\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{r}},\widetilde{R}_m^a) - \lambda_{\mathrm{r}}\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{r}},\widetilde{R}_m^r) - \boldsymbol{\delta}\mathbf{Cov}(\widetilde{\mathbf{P}}_{\mathbf{r}},\widetilde{\mathbf{S}}) - \boldsymbol{\pi}}{1+r},$$

where $\widetilde{\mathrm{V}}_{\mathrm{a}} = \widetilde{\mathbf{P}}_a\Sigma\mathbf{X}_a^*$, $\widetilde{\mathrm{V}}_{\mathrm{r}} = \widetilde{\mathbf{P}}_r\Sigma\mathbf{X}_r^*$, $\lambda = 1/(\Sigma\frac{-J_W}{J_{WW}}) > 0$, $\boldsymbol{\delta} = (\Sigma\frac{J_{WS}}{J_{WW}})/(\Sigma\frac{-J_W}{J_{WW}})$, $\lambda_{\mathrm{a}} = \lambda\mathbf{P}_a^0\Sigma\mathbf{X}_a^* = \lambda V_a^0$, $\lambda_{\mathrm{r}} = \lambda\mathbf{P}_r^0\Sigma\mathbf{X}_r^* = \lambda V_r^0$, $\boldsymbol{\pi} = \left[\Sigma[\mathbf{B}(2\mathbf{X}_r^* - \mathbf{q}_r)\Big/(\frac{1}{J_{WW}})]\right]\Big/(\Sigma\frac{J_W}{J_{WW}})$,

$\tilde{R}_m^a = (\tilde{V}_a - V_a^0)/V_a^0$, $\tilde{R}_m^r = (\tilde{V}_r - V_r^0)/V_r^0$, and $\mathbf{Cov}(\cdot)$ is the covariance operator.

Dividing Equations (18) and (19) by the current price $\mathbf{P_a^0}$ or $\mathbf{P_r^0}$ and re-arranging the terms, the two equations can be re-written as:

$$\tilde{\mathbf{R}}_a = \mathbf{r} + \lambda_a \mathbf{Cov}(\tilde{\mathbf{R}}_{\mathbf{a}}, \tilde{R}_m^a) + \lambda_r \mathbf{Cov}(\tilde{\mathbf{R}}_{\mathbf{a}}, \tilde{R}_m^r) + \boldsymbol{\delta}\mathbf{Cov}(\tilde{\mathbf{R}}_{\mathbf{a}}, \tilde{\mathbf{S}}), \tag{20}$$

and

$$\tilde{\mathbf{R}}_{\mathbf{r}} = \mathbf{r} + \lambda_a \mathbf{Cov}(\tilde{\mathbf{R}}_{\mathbf{r}}, \tilde{R}_m^a) + \lambda_r \mathbf{Cov}(\tilde{\mathbf{R}}_{\mathbf{r}}, \tilde{R}_m^r) + \boldsymbol{\delta}\mathbf{Cov}(\tilde{\mathbf{R}}_{\mathbf{r}}, \tilde{\mathbf{S}}) + \boldsymbol{\eta}, \tag{21}$$

where $\eta_i = \pi_i / P_{r_i}^0$. η_i and π_i are the ith element of $\boldsymbol{\eta}$ and $\boldsymbol{\pi}$, respectively. Equations (20) and (21) demonstrate that there are $k+2$ betas to generate the expected excess rate of return on financial and real estate assets. They are k betas for the k state variables, a financial market beta (related to the financial market rate of return $\tilde{R}_m^a$), and a real estate market beta (related to the real estate market rate of return $\tilde{R}_m^r$). However, for real estate assets, the effects of short-sale and limited-supply restrictions on asset pricing are determined by $\boldsymbol{\eta}$.

The Effect of the Constraints

We note that Equations (20) and (21) differ only in the pricing factor $\boldsymbol{\eta}$. That is, holding everything else constant, the short-sale and limited-supply constraints on the real estate asset are the two factors that can cause its return to differ from the financial asset return. A real estate asset can demand a higher (or lower) return than a financial asset if $\boldsymbol{\eta}$ is positive (or negative). If $\boldsymbol{\eta} = 0$, then the asset-pricing model for a financial asset should be the same as that for a real estate asset. Given this, it is important to analyze the sign of $\boldsymbol{\eta}$ under different conditions.

To do this, we first note that $\boldsymbol{\eta}$ is a summation of $\mathbf{B}(2\mathbf{X}_r^* - \mathbf{q}_r)/\frac{1}{J_{WW}}$ divided by the current price $\mathbf{P_r^0}$ and $\Sigma \frac{J_W}{J_{WW}}$. From the assumptions of increasing indirect utility of wealth $(J_W > 0)$ and decreasing marginal indirect utility of wealth $(J_{WW} < 0)$, $\Sigma \frac{J_W}{J_{WW}}$ must be < 0 and the sign of $\boldsymbol{\eta}$ must be determined by the sign of $\Sigma\mathbf{B}(2\mathbf{X}_r^* - \mathbf{q}_r)$. Equations (13) to (15) report that (1) $b_i = 0 = b_i(2X_{r_i}^* - q_{r_i})$, if $X_{r_i}^*$ is not binding, (2) $b_i(2X_{r_i}^* - q_{r_i}) = b_i q_{r_i} > 0$, if $X_{r_i}^*$ is binding at q_{r_i}, and (3) $b_i(2X_{r_i}^* - q_{r_i}) = -b_i q_{r_i} < 0$, if $X_{r_i}^*$ is binding at 0. Consequently, the sign of η_i (or the *i*-th element of η) must be a function of the Lagrangian multiplier b_i and the limited supply q_{r_i}. Given these, it is worthy to discuss the effect of the short-sale and limited-supply constraints on the asset-pricing model for real estate assets.

Limited-Supply Constraint

From Equation (14), we know that when the limited-supply constraint is binding, there must exist at least one investor such that $b_i(2X_{r_i}^* - q_{r_i}) = b_i q_{r_i} > 0$. Since $J_W > 0$ and $J_{WW} < 0$ for all investors, $b_i[(2X_{r_i}^* - q_{r_i}) / \frac{1}{J_{WW}}]/(\Sigma \frac{J_W}{J_{WW}})$ must be positive. The binding constraint implies that there would be excess demand in the market for the *i*-th real estate asset if the constraint were relaxed. This is true because under the limited-supply constraint, the total demand for the *i*-th real estate asset is $\Sigma X_{r_i}^* = \Sigma q_{r_i}$, and is exactly equal to the total available supply Q_{ri}. This means that at least one investor's optimal demand for the *i*-th real estate asset is at $X_{r_i}^* = q_{ri}$, while the rest of the investors' optimal demands for that real estate asset may be either non-negative for not binding, or zero for infeasible. (Under these circumstances, the total available *i*-th real estate asset becomes zero in Equation [9] for those investors.) The zero supply in the real estate market implies that the demand for the real estate asset must be zero or infeasible in the market. The binding condition $X_{r_i}^* = q_{ri}$ implies that this η_i is positive for the *i*-th real estate asset in Equation (3.13). This is because there is at least one investor whose $b_i(2X_{r_i}^* - q_{r_i}) = b_i q_{r_i} > 0$.

The binding condition happens when either the current price is too low (see Equation [19]) or the expected future price is too high. Under both scenarios, if there is no restriction on the available supply, the total optimal demand for this real estate asset would be more than the total supply in the market. The low current price or the high expected future price implies that the expected rate of return of the real estate asset should be higher than the rate of return when there is no restriction on the supply level of the property. Since the market clearing condition implies that the total demand must be equal to the total supply of all real estate asset *i*, the premium η_i for the limited-supply constraint for the real estate asset in Equation (21) must be positive.

We have frequently observed excess demand for real estate assets in many real estate markets. For example, in California, we observed excess demand for housing units in Los Angeles in the early 1980s and in San Francisco in the late 1990s. Similar situations occur in Asia, where the housing markets were extremely hot in the late 1980s in Taipei and in the early 1990s in Hong Kong. The condition of the excess demand normally persists for a period of time, partially because of the construction lag and liquidity problem in the real estate market. Under this limited-supply condition, since the current price is too low or the expected future price is too high, investors will have to bid a premium price over what is observed. The result of this section can be summarized in the following proposition.

Proposition I: If there are limited-supply restrictions on real estate assets and if such restrictions are binding, then the $k+2$ *asset pricing model derived with no limited-supply restrictions will under-estimate the expected excess rates of return of the real estate assets.*

No Binding Conditions

It is also possible that the $k+2$ model works for real estate assets. If an investor's optimal demand $X^*_{r_i}$ for the *i*-th real asset is exactly equal to the total supply $Q_{ri} = q_{ri}$ of the asset in the market, the available supply of this asset is reduced to zero after the transaction. That is, this asset becomes infeasible for the rest of the investors in the market. This will result in $\eta_i = 0$ in Equation (21) for the real estate asset *i*. Under this circumstance, the real estate market is in equilibrium for the *i*-th real estate asset and the pricing of real estate assets will follow the two-beta model (besides the $k-$betas representing the state variables). Indeed, if there is no restriction on the short sale (or limited supply), we can set **B = 0** (where **0** is a zero vector) in Equation (11), and the resulting asset-pricing model in Equation (21) will become an $k+2$ asset-pricing model. In other words, under this condition, our asset-pricing models in Equations (20) and (21) will be collapsed into the $k+2$ intertemporal asset-pricing model. The result can be summarized in the following proposition.

Proposition II: If the short-sale and limited-supply restrictions are not binding, then the expected excess rate of return on assets (financial or real estate assets) depends solely on the $k+2$ *betas: (i) the security market beta* *(the covariance of asset return with the security market return), (ii) the real estate beta (the covariance of asset return with the real estate market return), and the k-betas caused by the state variables. The* $k+2$ *beta asset-pricing model will be an unbiased estimator of the expected excess rate of return for both financial and real estate assets.*

Short-Sale Restriction

If the short-sale restriction is binding and the aggregation of the individual optimal demand for a real estate asset i is less than the total supply, then this asset is in excess supply and the market is not in a clearing condition. However, if short sales were allowed, the optimal demand would be less than zero and b_i would be zero. The binding of the no short-sale (i.e. $X^*_{r_i}=0$) condition on real estate asset i for some investors implies that the η_i must be negative in Equation (21) for real estate asset i. This is true because $b_i(2X^*_{r_i}-q_i)=-b_i q_i<0$ for some investors.

This condition appears when the current price of the real estate asset is too high, or the expected future price is too low. Both scenarios could trigger a short sale of that asset if there is no restriction on short sales. Equation (19) shows that the current price of the real estate asset will be higher than the price when there is no such restriction. Similarly, the expected excess rate of return on the real estate asset (Equation [21]) can be lower than the rate of return if the short-sale restriction is not effective. It should be noted that if the short-sale restriction is binding and if the market is not in equilibrium for the real estate asset *i*, the current price of the asset must be adjusted to be lower, or the expected future price of the asset must be adjusted to be higher than the current level in order to achieve the equilibrium condition. How long this adjustment process lasts will depend on the efficiency of the real estate market.

Those who have experienced the boom markets in Los Angeles (in the early 1980s), San Francisco (in the late 1990s), Taipei (in the late 1980s), and Hong Kong (in the early 1990s) can understand how slow it can be for the price level to drop from its peak. Although it might be difficult to clearly state the speed of adjustment of the price level, we can safely say that it is much slower than that observed in the financial markets. Indeed, under the no short-sale condition, investors will not be willing to pay for a property at the current price level since the expected future price is too low. The result of this section can be summarized in the following proposition.

Proposition III: If there are short-sale restrictions on real estate assets and if such restrictions are binding, the $k+2$ *beta model derived with no short-sale* *constraints will over-estimate the expected excess rate of return of real estate assets.*

Conclusions

In recent years, researchers have spent considerable efforts to analyze the risk and return relationship of Asian real estate investments using the CAPM or its variants. While it is true that similar models have been used extensively in the U.S. to study the performance of real estate investments, it might not be appropriate (as least for now) to apply such models directly to real estate investments in Asia. This is because one of the most important assumptions behind the CAPM and its variants is that investors can take unlimited short positions in any investments. While this assumption is violated to some degree in the U.S. property market, it is fully violated in Asian property markets.

In this paper, we showed that when investors cannot short sell properties, the results derived from asset pricing models can be biased. The intuition is simple. When a property market is too hot, investors cannot short sell the properties to bring the price into equilibrium. When there is excess demand in the market, and because no one can short sell the properties, the unfilled demand will push prices up. Under either scenario, it will take a long time for the market to adjust to equilibrium. Our paper explicitly addresses the conditions under which the CAPM's results are biased, and reports the parameters affecting the magnitudes of the biases.

It should be noted that the $k+2$ factor model developed in this paper is not intended to be an equilibrium model. The only purpose of our model is to point out the biases (in terms of magnitude and direction) of the CAPM when applied to a market that is not in equilibrium. We suggest that researchers in the future develop an equilibrium asset-pricing model that explicitly incorporates the short-sale constraint. This task, however, will not be easy. Otherwise, Ross (1977) would have developed an equilibrium model back then and would not have to only use a numerical example to demonstrate the impact of the short-sale constraint on the predictability of the CAPM.

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