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Pricing Mortgage-Backed Securities — First Hitting Time Approach[1]

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This paper develops a pricing model and derives a closed-form formula for valuing mortgage-backed securities (MBSs) that embed a barrier option feature while the optimal prepayment or refinancing choices of borrowers are endogenously determined. Given that "real estate investors" tend to prepay a loan relentlessly, an MBS with a high concentration of investor borrowers implies a lower MBS value. We specify the prepayment behavior of borrowers by using the first hitting time as a proxy for the trigger point of prepayment when house prices or interest rates hit a pre-determined barrier. Our results show that the MBS value is positively related to loan to value and house price volatility while negatively related to the proportion of real estate investors and interest rate volatility. We also find evidence which shows that the MBS value may increase due to the effects of the "longevity" of mortgages, which outweigh the effects of default or prepayment as house price volatility increases. This model provides a faster pricing tool of MBSs than Monte Carlo simulation while retaining higher model accuracy and consistency than the hazard model approach.

Keywords

Mortgage-Backed Security, First Hitting Time, Prepayment, Default

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1. Introduction

This paper develops a pricing model and derives a closed-form formula for valuing mortgage-backed securities (MBSs) that embed a barrier option feature while focusing on the prepayment aspects of MBSs. We use the first hitting time (FHT) as a proxy for the trigger point of prepayment as our logical approach to solving the valuation problem. The unique perspective of our study is our hypothesis that "real estate investors" among all borrowers behave differently from "regular borrowers": real estate investors are more likely to prepay loans.[3] Examining the effect of the proportion of real estate investors to all borrowers on MBS value is the focal point of our study.

Our contribution is threefold: (1) to the best of our knowledge, our paper is one of few studies to apply the concept of barrier option valuation to price MBSs and provide a model by using FHT to arrive at a true value rather than the confidence interval obtained by simulation; (2) we provide evidence to show that the proportion of real estate investors to all borrowers is a determining factor while valuing MBSs, and that the concentration of investor borrowers affects agency and non-agency MBS values differently; and (3) we examine and quantify the marginal effects on the MBS value of four factors that affect either the amount of cash payment from borrowers or the trigger timing of prepayment.

Depending on whether house prices or interest rates hit the pre-determined barriers, we examine four possible scenarios of payment choices of borrowers: (1) some borrowers prepay (but only "real estate investors" prepay); (2) all borrowers prepay; (3) default; and (4) holding onto the loan until maturity. After specifying the cash payments from the borrowers in each scenario, we apply the FHT model to determine the timing of the possible payment actions mentioned above and derive a closed-form formula for non-agency and agency MBS values. For robustness testing, we compare the pricing difference between our closed-form solution and Monte Carlo simulation in valuing MBSs.

Pricing MBSs is an important and challenging issue for both practitioners and academics. MBSs represent a claim on the cash payments from mortgage loans through a process known as securitization. Although any type of mortgage loan can be used as MBS collateral, most are backed by residential mortgages. Typical mortgages may have a term as long as 30 years, but quite often mortgages are paid off much sooner. Refinancing, foreclosure or house sales may be the reason for these unscheduled prepayments. In this paper, we focus on the prepayment aspects of MBSs and propose an MBS pricing model by

[3] This study mainly states that some real estate investors will sell their house in advance to earn a profit when they think that the house price is high. Normally, owner-occupiers will not sell their house when the house price is high. Alternatively, they will choose to rent a house then buy one when the price is low. Therefore, under the same conditions, real estate investors will have a higher propensity to prepay when the house price is high.

using the FHT model to identify the timing of the payment choices of borrowers. After specifying the cash payments in the possible scenarios from the prepayment of borrowers, we calculate the corresponding event probabilities and derive a closed-form pricing solution for MBS values accordingly.

Valuing MBS risk has become an increasingly important issue. The massive insurance of non-agency MBSs occurred from 2001 through to 2007 and then almost vanished in 2008 following the financial crisis.[4] In September 2012, the US Federal Reserve launched a third round of quantitative easing operations (QE3) which involved purchasing $40 billion USD of agency MBSs per month.[5] Compared to agency MBSs, non-agency MBSs lack government backing and incur additional credit risk. Other than that, there may be other factors that can influence both values differently. In this paper, we examine the effects of the proportion of "real estate investors" among all borrowers on both agency and non-agency MBS values. Due to differences in borrowing incentive, we hypothesize that "real estate investors" among all borrowers are more likely than "regular borrowers" to prepay loans and a negative effect of the proportion of "real estate investors" on MBS value is expected.

In other words, we endogenize the proportion of real estate investors to all borrowers in our pricing model which differentiates our paper from the previous literature on MBSs. In this study, our pricing model could endogenize the so-called taking out of equity, allow real estate investors to maximize their wealth and find the particular optimal time of taking out equity. However, since most real estate investors are still risk averters, maximizing utility is their ultimate goal. To avoid complicating the model, it is assumed that real estate investors have determined the optimal criterion for taking out equity based on their risk preference and then implement the criterion into the model. MBSs are collateralized by a pool of mortgage loans which are taken by "regular borrowers" and "real estate investors" for the purpose of self-dwelling and investment, respectively. Real estate investors can be expected to sell the collateralized houses or prepay the loans ruthlessly to maximize their investment return. For either profit-taking or loss-cutting, real estate investors may prepay the loans or sell the house earlier than regular borrowers. Our main focus is to examine the effect of the proportion of real estate investors among all borrowers on MBS value. We also examine the effects of three factors on MBS value which affect either the amount of cash payments from borrowers or the trigger timing of prepayment including: (1) loan to value, (2) interest rate volatility, and (3) house price volatility.

The prepayment choices of borrowers directly impact MBS valuation. Prepayments by individual mortgage holders affect both the amount and timing

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[4] According to "Non-agency Mortgage-Backed Securities, Managing Opportunities, and Risks" by JP Morgan in 2010: "The outstanding balance of non-agency mortgages grew from roughly $600 billion at the end of 2003 to $2.2 trillion at its peak in 2007".
[5] The aim of the QE3 program was to boost employment and the housing market.

of cash payments for MBS purchasers. We specify the cash payments of borrowers in four scenarios and price MBSs with barrier option features. We apply the FHT model to determine the timing of possible prepayments and derive the cumulative probability of hitting a preset barrier of interest rate or house price. We then classify four scenarios of cash payments from borrowers to specify the amount of cash payments. Combining the expected value of discounted future cash payment and their corresponding probabilities, we derive a closed-form formula for non-agency and agency MBS values.[6] For robustness testing, we compare the pricing difference between our formula and the Monte Carlo simulation in valuing MBSs.

The rest of the paper is organized as follows. The next section reviews the relevant literature that uses FHT to value exotic options or price MBSs. Section 3 introduces our MBS pricing model. Section 4 compares the simulation results with the results obtained from our pricing formula. Section 5 concludes.

2. Literature Review

Valuing MBSs is similar to valuing a set of real estate loans because both depend on the discounted value of future cash flows. As Barthèlèmy and Prigent (2009) point out, the value of a set of real estate loans equals to the sum of the discounted operating free cash flows during its anticipated holding period plus the discounted expected terminal value. The MBS value is calculated in a similar way, but depends on the discounted cash payments from the underlying mortgage loans. After specifying the cash payments by borrowers, we sum up the expected present values of future cash payments in each scenario and obtain the closed-form formula for MBS values.

Although the option approach to pricing MBS dates back to Dunn and McConnell (1981) who treat MBS as a callable bond, our paper is one of few studies to apply the concept of barrier option valuation to price MBSs while using the FHT model. We treat FHT as the time to retire or exercise the option embedded in MBSs and derive FHT density in Section 3.2 for the processes of interest rate and house price separately when either hits a pre-determined barrier. We then use that information to calculate the expected present value of future cash payments and obtain the closed-form formula for MBS values.

Lo, Chung and Hui (2007) propose a simple and easy-to-use method for calculating an accurate estimate of a double barrier hitting time distribution of mean-reverting lognormal processes. They discuss the application of their method to price exotic options with payoffs that are contingent on barrier hitting

6 Agency MBSs are issued by government-sponsored enterprises (GSEs) such as Ginnie Mae, Fannie Mae or Freddie Mac while non-agency MBSs are sponsored by private companies.

times. Lin (1998) used the method in Gerber and Shiu (1994, 1996) and Laplace transforms to derive the hitting time density with a double barrier. Lo and Hui (2006) derive a closed-form formula for the first passage time density of a time-dependent Ornstein-Uhlenbeck process. We adopt the probability density functions of the FHT for the processes of housing prices and interest rates, derived by Lin (1998) and Lo and Hui (2006).

Prepayment modeling is crucial to analyses of MBSs. Richard and Roll (1989) point out that prepayment may be triggered by (1) refinancing incentive, (2) seasonality (month of the year), (3) seasoning (age of the mortgage), and (4) burnout.[7] We add "type of borrower" as another attributing factor for prepayment. Our paper differs from the previous literature on MBSs in three ways. First, we apply the FHT model on the prepayment issue towards mortgage loans to determine the termination time of loans and price MBSs accordingly. Second, since we assume that the lower bound of housing prices, $L=UPB_t$ (unpaid balance at time *t*) is time-varying, we need to adjust the FHT density of the house price process unlike the constant lower barrier in Lin (1998) and Lo and Hui (2006). Third, we hypothesize that real estate investors among all borrowers are more likely to repay their loans than regular borrowers due to the differences in incentive. Hence, examining the effect of the proportion of real estate investors to all borrowers on MBS value is another focal point of our paper. To the best of our knowledge, we find few in the literature that have focused on this matter.

3. Methodology

In this section, we derive a closed-form formula for non-agency and agency MBS values based on four possible scenarios of payment choices of borrowers: (1) some borrowers prepay (but only "real estate investors" prepay); (2) all borrowers prepay; (3) default; and (4) holding onto the loan until maturity. We assume that housing prices and interest rates follow the geometric Brownian motion and mean reversion process, respectively. We treat the FHT as the time to exercise the embedded option, derive the FHT density, and price the MBS accordingly. The derivation consists of the following steps.

First, based on Lin (1998) and Lo and Hui (2006), we specify the cumulative density functions of the FHT for the processes of housing prices and interest rates. Second, in any given month, we specify the cash payments from borrowers in the four prepayment scenarios mentioned above. Third, we adjust the FHT density of the house price process since we assume that the lower bound of housing prices is time-varying. Then, we derive the adjusted

[7] Burnout describes a period of time over which fewer borrowers prepay their loans than expected despite reduced interest rates. The reasons include the lack of equity of some borrowers in the property or a reduction in their personal creditworthiness.

cumulative probabilities of house price hitting a house price barrier, (G_U^{adj}, G_L^{adj}) which are different from those (G_L , G_U) derived by Lin (1998). Fourth, we compute the expected present value of future cash payments in each scenario. Finally, we sum up the expected present values and obtain the closed-form formula for non-agency and agency MBS values.

3.1 Stochastic Processes of Housing Prices and Interest Rates

For the Brownian motion process, W_t, we define the FHT as $\tau_m = \min\{t \geqq 0, W_t = m\}$ if W_t reaches a threshold $m \in \Re$, and $\tau_m = \infty$ if W_t never hits m. We assume that house prices follow a geometric Brownian motion process:[8]

$$dH_t = \mu_h H_t dt + \sigma_h H_t dW_{ht} \tag{1}$$

where μ_h is the expected growth rate of house prices, σ_h is the volatility of the growth rate of house prices, and dW_{ht} is a standard Wiener process. Interest rates are assumed to follow a mean reversion (Ornstein-Uhlenbeck) process:

$$dr_t = \theta(\mu_r - r_t)dt + \sigma_r dW_{rt} \tag{2}$$

where $\mu_r$is the average weighted level of the interest rate process, and θ is the mean reversion speed which indicates the strength of the "attraction" described in μ_r. For small values of θ, the effect of this "attraction" disappears. For large values of θ, r_t would quickly converge to μ_r. Meanwhile, σ_r is the volatility of interest rates, and dW_{rt} is a standard Wiener process.

3.2 Probability Density Function of First Hitting Time

Based on Lin (1998) and Lo and Hui (2006), we specify the cumulative density functions of the FHT for the processes of housing prices and interest rates, respectively. The details are as follows.

3.2.1 First Hitting Time Density of Interest Rate Process

We assume that interest rates follow a mean reversion (Ornstein-Uhlenbeck) process. We denote τ_r as the FHT to the lower barrier of interest rates (r_l):

$$\tau_r = \inf\,\{t;\ r(t) = r_l,\ r_l < r(s)\ for\ all\ s \in [0, t)\}.$$

[8] There has been a long history of empirical studies that indicate an autocorrelated model is a better fit for the process. However, our study still leverages house price to follow a random process which is in line with most of the assumptions of real estate theoretical studies.

Lo and Hui (2006) provide a Fokker-Planck equation that is associated with a time-dependent Ornstein-Uhlenbeck process as follows:

$$\frac{\partial P}{\partial t}=\frac{1}{2}\sigma_r^2\frac{\partial^2 P}{\partial r^2}-\left[u(t)r+v(t)\right]\frac{\partial P}{\partial r}-u(t)P \tag{3}$$

$$\frac{\partial P}{\partial t}=\theta\frac{\partial}{\partial r}\left[\left(r-\mu_r\right)P\right]+\frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2}=\theta P+\theta r\frac{\partial P}{\partial r}-\theta\mu_r\frac{\partial P}{\partial r}+\frac{\sigma_r^2}{2}\frac{\partial^2 P}{\partial r^2} \tag{4}$$

From Equations (3) and (4), and $u(t)=-\theta v(t)=\theta\mu_r$, we can obtain the cumulative probability of hitting the lower barrier of interest rates at time t, $P_{fp}(r_0,t)$, as

$$\begin{aligned}P_{fp}(r_0,t)&=1-\int_{-\infty}^{r^*(t)}\left\{K(r,t;r_0,0)-K(r,t;-r_0,0)e^{-2\beta r_0}\right\}dr\\&=\Phi\left(\frac{2\beta\eta(t)+r_0}{\sqrt{2\eta(t)}}\right)+\Phi\left(-\frac{2\beta\eta(t)-r_0}{\sqrt{2\eta(t)}}\right)e^{-2\beta r_0}\end{aligned} \tag{5}$$

where $\Phi(.)$ is the cumulative normal distribution function, r_0 is the initial interest rate and

$$K(r,t;r',0)=\frac{1}{\sqrt{4\pi\eta(t)}}\exp\left\{-\frac{\left[re^{\alpha(t)}+\gamma(t)-r'\right]^2}{4\eta(t)}+\alpha(t)\right\}$$

where $\alpha(t)=-\int_0^t u(t')dt'=-\int_0^t -\theta dt'=\theta t$,

$$\gamma(t)=-\int_0^t v(t')e^{\alpha(t')}dt'=-\int_0^t \theta\mu_r e^{\theta t'}dt'=\mu_r\left(1-e^{\theta t}\right),$$

$$\eta(t)=\int_0^t\frac{1}{2}\sigma_r^2e^{2\alpha(t')}dt'=\int_0^t\frac{1}{2}\sigma_r^2e^{2\theta t'}dt'=\frac{\sigma_r^2}{4\theta}\left(e^{2\theta t}-1\right)$$

and $r^*(t)\equiv-\left[\gamma(t)+2\beta\eta(t)\right]e^{-\alpha(t)}$ is the $r(t)$ hitting the lower barrier r_l at time $t\geq 0$. Here, β is an adjustable parameter and has an optimal value:

$$\beta_{opt}=-\frac{\int_0^\tau \gamma(t)\eta(t)e^{-2\alpha(t)}dt}{2\int_0^\tau \eta^2(t)e^{-2\alpha(t)}dt}$$

where τ denotes the time at which the solution of the Fokker-Planck equation is evaluated. Given the barrier (r_l) and an initial interest rate(r_0), we obtain the cumulative probability of interest rates hitting a barrier at time t, $P_{fp}(r_0,t)$ as Equation (5).

3.2.2 First Hitting Time Density of Housing Price Process

We assume that housing prices follow a geometric Brownian motion. Following Lin (1998), we use double-barrier hitting time distributions in the housing price process $\{H(t)\}$. We allow L and U to be the lower and upper barriers for $\{H(t)\}$ with $0 < L < H < U$, respectively. Then we denote τ_L and τ_U as the FHT to the lower barrier (without hitting the upper barrier earlier) and the upper barrier (without hitting the lower barrier earlier), respectively. The purpose of setting an upper limit of house prices is that some real estate investors will sell the house in advance to gain profit when house prices appreciate. This is essentially different from a cash-out prepayment because a cash-out refinance allows the borrower to convert home equity into cash by creating a new mortgage for a larger amount than the original amount. On the other hand, the real estate investor is willing to default if the house price drops below a certain price level.

$$\tau_l = \inf\{t; H(t) = L, L < H(s) < \infty \text{ for all s} \in [0,t)\}$$
$$\tau_u = \inf\{t; H(t) = U, 0 < H(s) < U \text{ for all s} \in [0,t)\}$$

Then, the corresponding probability density functions are:

$$g_L(t;H_0,L,U) = \left[\frac{L}{H_0}\right]^{\left(\mu_h\sigma_h^{-2}-\frac{1}{2}\right)} \exp\left\{-\frac{1}{2}\left(\mu_h\sigma_h^{-1}-\frac{1}{2}\sigma_h\right)^2 t\right\} \sum_{n=-\infty}^{\infty}\frac{a_n}{\sqrt{2\pi t^3}}e^{-\frac{a_n^2}{2t}} \tag{6}$$

$$g_U(t;H_0,L,U) = \left[\frac{U}{H_0}\right]^{\left(\mu_h\sigma_h^{-2}-\frac{1}{2}\right)} \exp\left\{-\frac{1}{2}\left(\mu_h\sigma_h^{-1}-\frac{1}{2}\sigma_h\right)^2 t\right\} \sum_{n=-\infty}^{\infty}\frac{b_n}{\sqrt{2\pi t^3}}e^{-\frac{b_n^2}{2t}} \tag{7}$$

where $a_n = \frac{1}{\sigma_h}\ln\frac{U^{2n}H_0}{L^{2n+1}}$, $b_n = \frac{1}{\sigma_h}\ln\frac{U^{2n+1}}{L^{2n}H_0}$ and H_0 is the initial house price. We take the integrals of g_L and g_U and obtain the cumulative density functions G_L and G_U as Appendix I exhibits.

Given the barriers (L, U) and an initial house price (H_0), the cumulative distribution functions $G_L(t;H_0,L,U)$ and $G_U(t;H_0,L,U)$ can be explained as the cumulative probability of house prices hitting the lower and upper bounds at time t, respectively.

3.3 Four Scenarios of Cash Payments from Borrowers

Cash payments from borrowers are vital for MBS valuation. At each point in time during the amortization period, borrowers could choose from one of the following payment actions: (1) make a complete prepayment of the loan, (2) default, or (3) continue to hold onto the loan until maturity (pay the scheduled payment in full).[9] In this section, we analyze the cash payments from borrowers based on four scenarios: (1) house prices (H_t) hit the upper bound; (2) interest rates (r_t) hit the lower bound and all borrowers prepay; (3) H_t hit the lower bound; and (4) H_t and r_t do not hit any barriers.

In Scenario (1), when house prices are high enough, we assume that only real estate investors will prepay their loan to cash out on the significant increase in equity price and look for the next investment opportunity.[10] We assume default happens in Scenario (3) when house prices reach the lower bound, which is the most common definition found in the related literature. When default happens, the cash payments of the borrowers to MBS purchasers are different for non-agency and agency MBSs. We use Scenario (3) to differentiate between non-agency and agency MBSs. When interest rates are low enough, that is, Scenario (2), we assume that all borrowers refinance and prepay the loan. As for Scenario (4), nothing happens and nothing changes. We allow r_l to be the lower bound of interest rates, and (U, L) the upper and lower bounds of house prices, respectively. We then analyze the cash payments from borrowers in each scenario as follows.

Scenario (1) House prices (H_t) hit upper bound $U=(1+a)H_0$

When house prices are high enough and hit a preset upper bound, real estate investors may prepay their loan to cash in and look for another investment opportunity which may not be the case for regular borrowers. We assume that real estate investors among all of the borrowers will make a complete prepayment of the loan when H_t hit a preset upper bound $U= H_0(1+a)$, where H_0 is the initial house price and a is an arbitrary constant. The cash payment at time t (CF_{1t}) is equal to the mortgage payment (*PMT*) plus the unpaid balance.

$$CF_{1t} = PMT + UPB_t$$

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[9] As the definition used in most of the related literature, we define the occurrence of *default* when the house price is less than the unpaid balance at time t ($H_t < UPB_t$) and that of *prepayment* when the expected present value of future cash payments is larger than UPB_t (or UPB_t plus transaction cost).

[10] Real estate investors may take out a home equity line of credit (HELOC) to cash out on the significant increase in equity price and do not need to sell the home. We thank an anonymous referee for his/her advice.

Scenario (2) Interest rates (r_t) hit lower bound r_l

When interest rates are low enough and hit a preset lower bound, we assume that all borrowers will refinance their purchase and prepay the loan.[11] Refinancing loans results in paying off existing mortgages before the expected maturity date of the loans. The cash payment at time t (CF_{2t}) is then equal to the mortgage payment (*PMT*) plus the unpaid balance (UPB_t):

$$CF_{2t} = PMT + UPB_t$$

Scenario (3) House prices (H_t) hit lower bound $L = UPB_t$

As is the case in most of the related literature, we assume that default occurs when the house prices are low enough and hit a preset lower bound $L=UPB_t$. In reality, borrowers may not default and tend to continue to pay their mortgage even when $H_t \leqq UPB_t$ because some transaction costs have taken place.[12] However, allowing $L = UPB_t$ will simplify our scenario analysis without loss of generality. We discuss the cash payments in the event of default for non-agency and agency MBSs as follows.

a. For non-agency MBSs, the private companies who sponsored the MBS issuance will regard house prices at time t (H_t) as the unpaid balance and pay the house price to MBS purchasers when H_t hit the lower bound L. Hence, the cash payment at time t is equal to the house prices at time t (H_t): $CF_{3t}^a = H_t$

b. For agency MBSs, the government-sponsored enterprises who issued the MBS under consideration will pay UPB_t to the MBS buyers when H_t hit the lower bound L and own the right to dispose the mortgaged house. Therefore, the cash payment at time t is equal to the unpaid balance at time t (UPB_t): $CH_{3t}^b = UPB_t$

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[11] When the interest rate is below a certain threshold, there is the willingness to borrow new debt and repay the old ones. This number may be endogenous. In terms of net present value without considering the risk premium, as long as the market interest rate is lower than the contract rate, there is a willingness to refinance. However, if there is the desire to transfer loans by "borrowing new and repaying the old", the fees associated with prepayment must be paid. Therefore, each mortgage borrower usually has an interest rate in mind that is lower than the contract rate to a certain extent. This spread will differ from person to person, so our model uses this spread to conduct a comparative static analysis. In addition, in terms of of low interest rates and high house prices, we assume that the boundaries for interest rates and house prices which are hit first will result in exercising the prepayment option.

[12] Due to transaction costs, the borrower tends to pay the mortgage even if they are slightly underwater. We can take transaction cost into consideration by setting the lower bound as $L_t = (1 - b)UPB_t$, where b is some threshold and $0 \leqq b < 1$. However, our closed form solution is still functional and we omit the change in this paper. We thank an anonymous referee for his/her advice.

Scenario 4 House prices (H_t) and interest rates (r_t) do not hit any barriers

We assume that all borrowers will continue to pay their mortgage until the next payment date when house prices (H_t) and interest rates (r_t) do not hit any barriers at time *t*. Hence, the cash payment at time *t* is equal to the mortgage payment (*PMT*): $CF_{4t} = PMT$

3.4 Adjusted Probability of Hitting House Price Barrier $\left(G_U^{adj}, G_L^{adj}\right)$

Previously, given constant barriers (*L*, *U*) and an initial house price (H_0), we obtain the cumulative probability of hitting the lower and upper bounds of house prices at time *t*, $G_L(t; H_0, L, U)$ and $G_U(t; H_0, L, U)$, respectively. However, since our lower bound of house price is the unpaid balance at time *t*, which is not a constant but time-varying, we need to adjust the FHT density of the house price process. In this section, we provide the adjusted cumulative probabilities of house prices hitting a house price barrier, (G_U^{adj}, G_L^{adj}) which are different from those (G_L , G_U) derived by Lin (1998) in Equations (A1) and (A2) from Appendix I. By letting $t_1 \leqq t_2 \leqq t_3 \leqq \ldots\ldots \leqq t_n$, we derive the adjusted cumulative probability of house prices hitting the upper bound (*U*) at time *t* (G_U^{adj}) as follows:

a. $G_U(t_1; H_0, L_1, U)$ remains the same and allows for $G_U^{adj}(t_1) = G_U(t_1; H_0, L_1, U)$.

b. $G_U(t_2; H_0, L_2, U)$ is overstated because if $L_2 \le H_1 \le L_1$ at time t_1, borrowers will default at time t_1. It is impossible for house prices to hit the upper bound *U* at time t_2, so we adjust $G_U(t_2)$ to:
$$G_U^{adj}(t_2) = G_U(t_2; H_0, L_2, U) - \left[G_U(t_1; H_0, L_2, U) - G_U(t_1; H_0, L_1, U)\right]$$

c. $G_U(t_3; H_0, L_3, U)$ should exclude the probabilities of $L_2 \le H_1 \le L_1$ at time t_1 and $L_3 \le H_2 \le L_2$ at time t_2. Thus, the adjusted $G_U(t_3)$ is:
$$\begin{aligned} G_U^{adj}(t_3) = {} & G_U(t_3; H_0, L_3, U) \\ & - \left[G_U(t_1; H_0, L_2, U) - G_U(t_1; H_0, L_1, U)\right] \\ & - \left[G_U(t_2; H_0, L_3, U) - G_U(t_2; H_0, L_2, U)\right] \end{aligned}$$

d. By deduction, we can obtain the adjusted $G_U^{adj}(t_n)$ where n = 2, 3, …, T, as
$$G_U^{adj}(t_n) = G_U(t_n; H_0, L_n, U) - \sum_{i=1}^{n-1}\left[G_U(t_i; H_0, L_{i+1}, U) - G_U(t_i; H_0, L_i, U)\right]$$
Similarly, we derive the adjusted cumulative probability of $\{H_t\}$ hitting the lower bound (*L*) at time *t*, (G_L^{adj}), as follows:

$G_L^{adj}(t_1) = G_L(t_1; H_0, L_1, U)$ and

$$G_L^{adj}(t_n) = G_L(t_n; H_0, L_n, U) + \sum_{i=1}^{n-1}\left[G_U(t_i; H_0, L_i, U) - G_U(t_i; H_0, L_{i+1}, U)\right]$$

We summarize the results of Sections 3.2 to 3.4 in Table 1.

3.5 Expected Present Value of Future Cash Payments and MBS Value

From Table 1, we have the cumulative probability of $\{H_t\}$ or $\{r_t\}$ hitting a barrier at time t, and the actions that borrowers will take if one of six scenarios occurs. To calculate the expected present value of future cash payments by borrowers, we need to know the probability of $\{H_t\}$ or $\{r_t\}$ hitting a barrier during [t-1/12, t], (ΔG_U, ΔG_L, ΔP_{fp}). The probability of $\{H_t\}$ or $\{r_t\}$ hitting a barrier during [t-1/12, t] is equal to the cumulative probability of hitting a barrier at time t minus the cumulative probability of hitting a barrier at time t-1/12 as shown in Table 2.

Table 1 Cumulative Probability of $\{H_t\}$ or $\{r_t\}$ Hitting Barrier at Time t

H_t / r_t	H_t hit U ($\tau_u < t$)	No hits	H_t hit L ($\tau_l < t$)	Cumulative Probability
No hits ($\tau_r > t$)	ρ (Investors only) prepay	All continue to pay loan	Default	1-$P_{fp}(r_0, t)$
r_t hit r_l ($\tau_r < t$)	All prepay	All prepay	Default	$P_{fp}(r_0, t)$
Cumulative Probability	$G_U^{adj}(t)$	$1 - G_U^{adj}(t) - G_L^{adj}(t)$	$-G_L^{adj}(t)$	1

$$\Delta G_U(t; H_0, L, U) = G_U^{adj}(t) - G_U^{adj}\left(t - \tfrac{1}{12}\right) \tag{8}$$

$$\Delta G_L(t; H_0, L, U) = G_L^{adj}(t) - G_L^{adj}\left(t - \tfrac{1}{12}\right) \tag{9}$$

$$\Delta P_{fP}(r_0, t) = P_{fP}(r_0, t) - P_{fP}\left(r_0, t - \tfrac{1}{12}\right) \tag{10}$$

From Table 2, the probability of $\{H_t\}$ hitting the upper or lower bound during [t-1/12, t] is $\Delta G_U(t; H_0, L, U)$ or $\Delta G_L(t; H_0, L, U)$, respectively. The probability of $\{r_t\}$ hitting the lower bound during [t-1/12, t] is $\Delta P_{fp}(r_0, t)$. From Table 1, the probability that house prices never hit any bounds at time t is $1 - G_U^{adj}(t) - G_L^{adj}(t)$ and the probability that the interest rates never hit the lower bound at time t is 1-$P_{fp}(r_0, t)$. Next, we derive the probability of each of the four scenarios mentioned above for each month, with (P_1, P_2, P_3, P_4) representing ($P\rho_{\text{prepay}}$, $P_{\text{all prepay}}$, P_{default}, P_{holding}), respectively.

Table 2 Probability of {H_t} or {r_t} Hitting a Barrier during [t-1/12, t]

H_t / r_t	H_t hit U ($\tau_u < t$)	H_t hit L ($\tau_l < t$)	Probability
r_t hit r_l ($\tau_r < t$)	All prepay	Default	$\Delta P_{fp}(r_0, t)$
Probability	$\Delta G_U(t; H_0, L, U)$	$\Delta G_L(t; H_0, L, U)$	--

If {H_t} hits the upper bound and {r_t} never hits the lower bound at time t, we assume that only a proportion of the borrowers (real estate investors) will choose to prepay their loans because of the opportunity of cashing in their investment. If {H_t} and {r_t} are mutually independent, the probability of partial prepayment (as in Scenario 1) can be written as:

$$P_1(t) \equiv P_{\rho\,\text{prepay}}(t) = \rho \times \Delta G_U(t; H_0, L, U) \times \left[1 - P_{fp}(r_0, t)\right] \tag{11}$$

When {r_t} hits the lower bound and {H_t} is above the lower bound, all borrowers will choose to prepay because of the relatively low cost of refinancing. The probability of all prepayment (as in Scenario 2) is:

$$\begin{aligned} P_2(t) &\equiv P_{\text{all prepay}}(t) \\ &= \left[\Delta G_U(t; H_0, L, U) + 1 - G_U^{adj}(t) - G_L^{adj}(t)\right] \times \Delta P_{fP}(r_0, t) \end{aligned} \tag{12}$$

Regardless of {r_t}, when {H_t} hits the lower bound at time t, we assume that all borrowers will choose to default because of the extremely low house price. Hence, the probability of default (as in Scenario 3) is:

$$P_3(t) \equiv P_{\text{default}}(t) = \Delta G_L(t; H_0, L, U) \times \left[1 - P_{fP}(r_0, t) + P_{fP}(r_0, t)\right] \tag{13}$$

If {H_t} and {r_t} do not hit any barriers at time t, we assume that all borrowers will choose to hold on and continue to pay their mortgage until the next payment date. Hence, the probability of holding (as in Scenario 4) is:

$$P_4(T) \equiv P_{\text{holding}}(T) = 1 - \sum_{t=1/12}^{T-1/12} \left[P_1(t) + P_2(t) + P_3(t)\right] \tag{14}$$

To calculate the expected present value of future cash payments by borrowers, we need a series of discount rates. Since {r_t} follows a Ornstein-Uhlenbeck process, the discount rate is random and we set $\mathrm{R(t)} = \int_0^t r_s ds$. Based on Equations (1) and (2), we apply Ito's Lemma and obtain Equations (15) and (16) as follows:

$$R(t) \sim N\left(\mu_r t + (r_0 - \mu_r)\frac{1-e^{-\theta t}}{\theta}, \frac{\sigma_r^2}{\theta^2}\left(t - \frac{1-e^{-\theta t}}{\theta} - \frac{1}{2}\theta\left(\frac{1-e^{-\theta t}}{\theta}\right)^2\right)\right) \quad (15)$$

$$H_t = H_0 \exp\left(\int_0^t r_s\, ds - \frac{1}{2}\sigma_h^2 + \sigma_h \sqrt{t} z\right) \quad (16)$$

If the mortgage rate is r_c and maturity T, we can easily obtain the expressions for mortgage payment (*PMT*) and unpaid balance at time t (UPB_t) as follows:

$$PMT = \frac{principle}{(e^{-\frac{r_c}{12}} + e^{-\frac{r_c}{12}\times 2} + e^{-\frac{r_c}{12}\times 3} + 3 + e^{-\frac{r_c}{12}\times 12T})} = principle \times \frac{1 - e^{\frac{r_c}{12}}}{e^{-\frac{r_c}{12}\times 12T} - 1}$$

$$UPB_t = PMT \times (e^{-\frac{r_c}{12}} + e^{-\frac{r_c}{12}\times 2} + e^{-\frac{r_c}{12}\times 3} + \ldots + e^{-\frac{r_c}{12}\times 12(T-t)})$$

$$= PMT \times \frac{e^{-\frac{r_c}{12}\times 12(T-t)} - 1}{11 e^{\frac{r_c}{12}}}$$

where $principal = LTV \times H_0$ and *LTV* is the loan to value and H_0 is the initial house price. We then discount the cash payments at time t in Section 3.3 to obtain the present value in the event of prepayment (PV_{1t}), default (PV_{2t}^a and PV_{2t}^b), and holding on until loan maturity (PV_{3T}) as follows. Recall that PV_{2t}^a is for non-agency MBSs and PV_{2t}^b for agency MBSs.

$$PV_{1t} = PMT \times (e^{-\int_0^{\frac{1}{12}} r_s\, ds} + e^{-\int_0^{\frac{2}{12}} r_s\, ds} + \ldots + e^{-\int_0^{t} r_s\, ds}) + UPB_t \times e^{-\int_0^{t} r_s\, ds}$$

$$PV_{2t}^a = PMT \times (e^{-\int_0^{\frac{1}{12}} r_s\, ds} + e^{-\int_0^{\frac{2}{12}} r_s\, ds} + \ldots + e^{-\int_0^{t-\frac{1}{12}} r_s\, ds}) + H_t \times e^{-\int_0^{t} r_s\, ds}$$

$$PV_{2t}^b = PMT \times (e^{-\int_0^{\frac{1}{12}} r_s\, ds} + e^{-\int_0^{\frac{2}{12}} r_s\, ds} + \ldots + e^{-\int_0^{t-\frac{1}{12}} r_s\, ds}) + UPB_t \times e^{-\int_0^{t} r_s\, ds}$$

$$PV_{3T} = PMT \times (e^{-\int_0^{\frac{1}{12}} r_s\, ds} + e^{-\int_0^{\frac{2}{12}} r_s\, ds} + \ldots + e^{-\int_0^{T} r_s\, ds})$$

Since we know that $\{r_t\}$ is stochastic, we derive the expectations of ($PV_{1t}, PV_{2t}^a, PV_{2t}^b, PV_{3T}$), denoted as ($V_{1t}$, V_{2t}^a, V_{2t}^b, V_{3T}), respectively. The derivations are provided in the Appendix II. We allow $V_{1t} = \mathrm{E}[\mathrm{P}V_{1t}]$, $V_{2t}^a = \mathrm{E}[PV_{2t}^a]$, $V_{2t}^b = \mathrm{E}[PV_{2t}^b]$, and $V_{3T} = \mathrm{E}[PV_{3T}]$. Then, we can obtain the values of the non-agency and agency MBSs as:

$$\text{Non-agency MBS value} = \sum_{t=1/12}^{T-1/12}\left[P_1(t)V_{1t} + P_2(t)V_{1t} + P_3(t)V_{2t}^a\right] + P_4(T)V_{3T} \quad (17)$$

Agency MBS value

$$= \sum_{t=1/12}^{T-1/12} \left[P_1(t)V_{1t} + P_2(t)V_{1t} + P_3(t)V_{2t}^{b} \right] + P_4(T)V_{3T} \tag{18}$$

4. Numerical Results and Discussion

In this section, we calculate the MBS values from Equations (17) and (18) and compare them with those from the Monte Carlo simulation for robustness testing. By using a sensitivity analysis, we also examine the effects of four factors that affect either the amount or timing of cash payments by borrowers on MBS value: (1) loan to value (*LTV*), (2) proportion of real estate investors (ρ), (3) interest rate volatility (σ_r), and (4) house price volatility (σ_h). The average impacts per 1% change of (*LTV*, ρ, σ_r) on both non-agency and agency MBS values obtained by our pricing formula are (+3.34%, -0.42%, -0.42%), respectively. However, the percentage change of non-agency and agency MBS values per 1% change of σ_h is +0.28% and 0.02%, respectively. The numerical results of the simulation and sensitivity analysis are provided below.

4.1 Parameter Calibration of Monte Carlo Simulation

To demonstrate the performance of Equations (17) and (18), we carry out a Monte Carlo simulation with 100,000 iterations and assume that housing prices follow a geometric Brownian motion process with μ_h=0.03 and σ_h=0.05.[13] Recall that we assume that interest rates follow an Ornstein-Uhlenbeck process with μ_r=0.04, θ=0.25 and σ_r=0.01.[14]

$$dH_t = \mu_h H_t dt + \sigma_h H_t dW_{ht} \rightarrow dr_t = \theta(\mu_r - r_t)dt + \sigma_r dW_{rt}$$

The parameters that we calibrate for the Monte Carlo simulation and our MBS pricing formula are listed in Table 3. Given those parameters, we calculate the values for non-agency and agency MBSs and compare them with the values from the Monte Carlo simulation. In Table 4, we find that the MBS value from our closed-form formula is slightly higher than that of the Monte Carlo simulation, and the difference is less than 0.1%. The results hold true for both non-agency and agency MBSs.

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[13] The parameters are imputed from Buist and Yang (1998), Yang *et al.* (2011), and Stephen *et al.* (1995). The value of μ_h is the average from Buist and Yang (1998) and Yang *et al.* (2011). The value of σ_h in Stephen *et al.* (1995) is 0.05, 0.1 in Buist and Yang (1998), and 0.02 in Yang *et al.* (2011). Therefore, we use the median of these for our simulation parameters.

[14] Regarding μ_r, we use the 30 year treasury rate for 2010. Chan *et al.* (1992), Buist and Yang (1998), Lin *et al.* (2006), and Yang *et al.* (2011) provide values for σ_r of 0.08, 0.03, 0.15, and 0.15, respectively. We choose 0.08 as the volatility measure. However, because the measures are based on the Cox–Ingersoll–Ross (CIR) model, we convert them into the model in Vasicek (1997), which we utilize. This conversion results in a value of 0.01 for our simulation parameter.

Table 3 Calibrated Parameters for MBS Pricing Formula

Parameter	Value
Initial House Price (H_0)	5,000,000
Initial Interest Rate (r_0)	0.02
House Price mean and Volatility ($\mu_h,\ \sigma_h$)	(0.03, 0.05)
Interest Rate mean and Volatility ($\mu_r,\ \sigma_r$)	(0.04, 0.01)
Mean Reversion Speed of Interest Rate (θ)	0.25
Loan to Value (*LTV*)	0.8
Coupon rate (r_c)	0.04
Proportion of Investors (ρ)	0.2
Upper bound constant (a) where $U= H_0(1+a)$	0.25
Maturity (T)	30 (years)

Table 4 MBS Values from Our Model vs. Monte Carlo Simulation

Non-Agency MBS			Agency MBS		
closed form	**simulation**	**error**	**closed form**	**simulation**	**error**
4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%

4.2 Effects of Loan to Value

Table 5 shows the relationship between loan to value (*LTV*) and MBS value. We find that the MBS value monotonically increases as the *LTV* increases from 30% to 80% in both the Monte Carlo simulation and our pricing model.[15] The MBS value from our closed-form formula is slightly higher than that of the Monte Carlo simulation, and the difference remains constant and less than 0.1% when the *LTV* increases. The average impact per 1% change of *LTV* on MBS value obtained by our pricing formula is +3.34%. The results hold true for both non-agency and agency MBSs. Figures 1 and 2 illustrate a positive relationship between *LTV* and MBS value for non-agency and agency MBSs, respectively.

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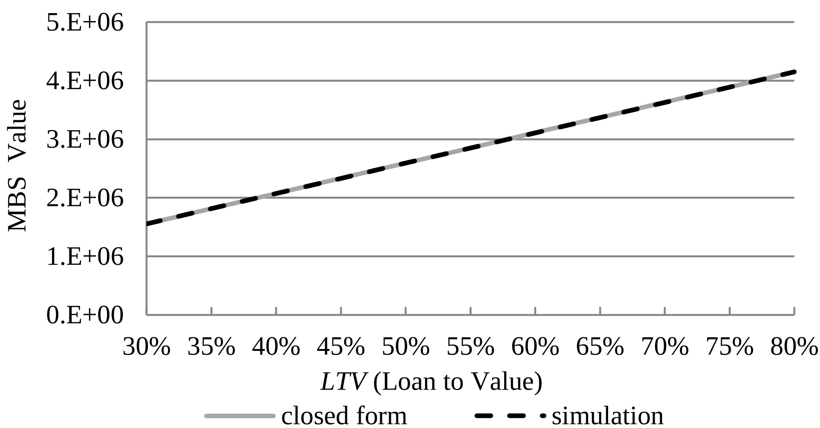
[15] For example, say that house price is fixed at 5 million. If *LTV*=80%, the MBS value should be around 4 million. Likewise, if *LTV*=60%, the MBS value should be around 3 million. As a result, a higher *LTV* results in higher MBS value, and so this is not economically counter-intuitive.

Table 5 Relationship between Loan to Value (*LTV*) and MBS value

	Non-Agency MBS			Agency MBS		
LTV	**closed form**	**simulation**	**error**	**closed form**	**simulation**	**error**
30%	1556079.14	1555042.75	0.07%	1556079.14	1555042.75	0.07%
35%	1815424.78	1814215.67	0.07%	1815424.78	1814215.67	0.07%
40%	2074769.16	2073387.33	0.07%	2074769.16	2073387.33	0.07%
45%	2334112.29	2332557.77	0.07%	2334112.29	2332557.77	0.07%
50%	2593456.69	2591729.51	0.07%	2593456.69	2591729.51	0.07%
55%	2852811.56	2850911.56	0.07%	2852811.56	2850911.56	0.07%
60%	3112202.85	3110129.05	0.07%	3112202.85	3110129.05	0.07%
65%	3371698.07	3369447.01	0.07%	3371698.06	3369447.00	0.07%
70%	3631469.32	3629031.22	0.07%	3631468.97	3629030.86	0.07%
75%	3891961.43	3889310.46	0.07%	3891954.23	3889303.27	0.07%
80%	4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%

Figure 1 Relationship between Loan to Value (*LTV*) and Non-Agency MBS Value

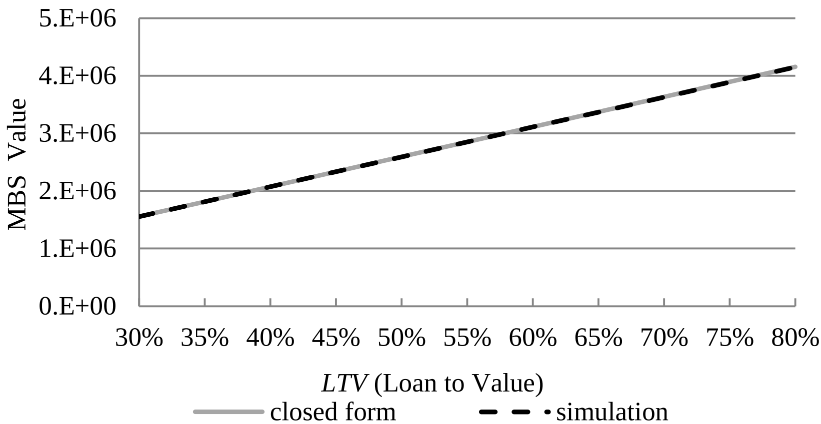


4.3 Effects of Proportion of Real Estate Investors

Table 6 shows the relationship between MBS value and the proportion of real estate investors among all borrowers (ρ). We find that the MBS value decreases as ρ increases from 10% to 20% in both the Monte Carlo simulation and our pricing model. The MBS value from our closed-form formula is slightly higher than that of the Monte Carlo simulation, and the difference remains constant and less than 0.1% when ρ increases. The average impact per 1% change of

ρ on the MBS value obtained by our pricing formula is -0.42%.[16] The results hold true for both non-agency and agency MBSs.

Figure 2 Relationship between Loan to Value (*LTV*) and Agency MBS Value



We expect that the probability of prepayment increases when the proportion of real estate investors among all borrowers increases. Prepayment reduces future cash payments by borrowers, and thereby reduces the MBS value. Our results are consistent with intuition. Figures 3 and 4 illustrate the negative relationship between ρ and the MBS values for non-agency and agency MBSs, respectively.

Table 6 Relationship between Proportion of Investors (ρ) and MBS Value

	Non-Agency MBS			Agency MBS		
ρ	**closed form**	**simulation**	**error**	**closed form**	**simulation**	**error**
0.10	4156121.93	4153132.57	0.07%	4156010.22	4153021.05	0.07%
0.11	4155948.65	4152965.22	0.07%	4155836.94	4152853.70	0.07%
0.12	4155775.37	4152797.88	0.07%	4155663.65	4152686.36	0.07%
0.13	4155602.09	4152630.53	0.07%	4155490.37	4152519.01	0.07%
0.14	4155428.81	4152463.19	0.07%	4155317.09	4152351.66	0.07%
0.15	4155255.53	4152295.84	0.07%	4155143.81	4152184.32	0.07%
0.16	4155082.25	4152128.49	0.07%	4154970.53	4152016.97	0.07%
0.17	4154908.97	4151961.15	0.07%	4154797.25	4151849.63	0.07%
0.18	4154735.69	4151793.80	0.07%	4154623.97	4151682.28	0.07%
0.19	4154562.41	4151626.46	0.07%	4154450.69	4151514.94	0.07%
0.20	4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%

[16] In the extreme case where ρ =100%, house prices could be reduced by as much as 42%.

4.4 Effects of Interest Rate Volatility

Table 7 shows the relationship between the MBS value and interest rate volatility (σ_r). We find that the MBS value decreases as σ_r increases in both the Monte Carlo simulation and our pricing model. The MBS value from our closed-form formula is slightly higher than that of the Monte Carlo simulation, and the difference becomes more substantial as σ_r increases. For instance, when σ_r increases from 0.1% to 1.5%, the difference in the MBS values from both approaches increases from 0.06% to 0.18% for non-agency MBSs. The average impact per 1% change of σ_r on the MBS value obtained by our pricing formula is -0.42%. The results hold true for both non-agency and agency MBSs.

Figure 3 Relationship between Proportion of Investors and Non-Agency MBS Value

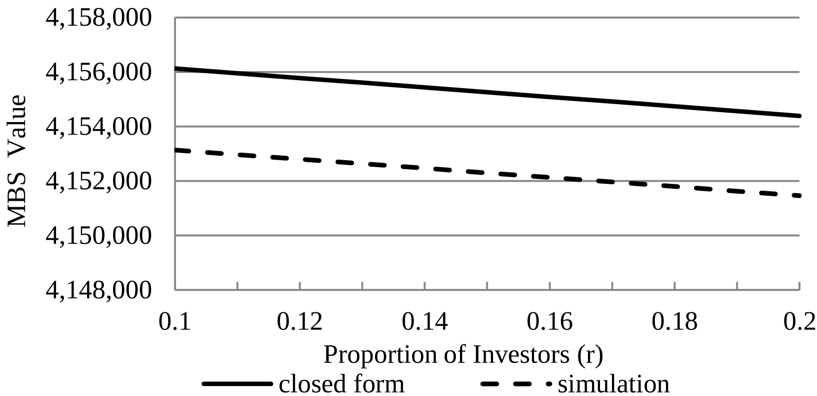
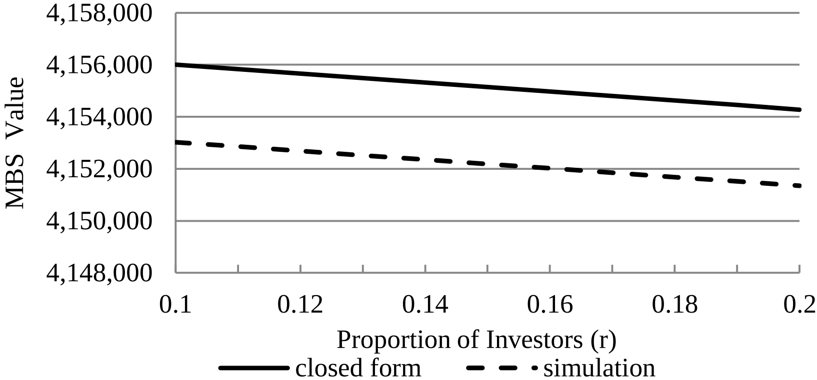


Figure 4 Relationship between Proportion of Investors and Agency MBS Value



It is easier for $\{r_t\}$ to hit the lower bound when interest rate volatility increases. If so, the probability of prepayment increases and holding onto the loan until maturity decreases, which will reduce future cash payments by borrowers, and thereby decrease the MBS value. Our results are consistent with intuition. Figures 5 and 6 illustrate the monotonic relationship between σ_r and the MBS values for non-agency and agency MBSs, respectively.

Table 7 Relationship between Interest Rate Volatility (σ_r) and MBS Value

	Non-Agency MBS			Agency MBS		
σ_r	closed form	simulation	error	closed form	simulation	error
0.1%	4169693.28	4167155.42	0.06%	4169605.69	4167067.95	0.06%
0.2%	4169212.22	4166626.32	0.06%	4169119.47	4166533.70	0.06%
0.3%	4168296.22	4165670.18	0.06%	4168198.48	4165572.59	0.06%
0.4%	4166945.28	4164287.00	0.06%	4166842.74	4164184.62	0.06%
0.5%	4165216.84	4162537.29	0.06%	4165110.41	4162431.04	0.06%
0.6%	4163213.48	4160523.98	0.06%	4163104.29	4160414.97	0.06%
0.7%	4161044.64	4158353.12	0.06%	4160933.71	4158242.38	0.06%
0.8%	4158805.06	4156102.92	0.07%	4158693.25	4155991.30	0.07%
0.9%	4156568.52	4153806.95	0.07%	4156456.50	4153695.13	0.07%
1.0%	4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%
1.1%	4152304.97	4149028.94	0.08%	4152193.94	4148918.12	0.08%
1.2%	4150341.92	4146480.90	0.09%	4150231.87	4146371.07	0.09%
1.3%	4148516.83	4143784.47	0.11%	4148407.97	4143675.85	0.11%
1.4%	4146839.95	4140917.32	0.14%	4146732.44	4140810.06	0.14%
1.5%	4145316.83	4137871.55	0.18%	4145210.78	4137765.78	0.18%

4.5 Effects of House Price Volatility

Table 8 shows the relationship between the MBS value and house price volatility (σ_h). We find that the MBS value increases as σ_h increases in both the Monte Carlo simulation and our pricing model. Again, the MBS value from our closed-form formula is slightly higher than that of the Monte Carlo simulation, and the difference is relatively constant across the σ_h. For instance, when σ_h increases from 1% to 10%, the difference in the MBS values from both approaches only increases from 0.07% to 0.08% for non-agency MBSs. The results hold true for both non-agency and agency MBSs. However, the average percentage change of the non-agency and agency MBS values per 1% change of σ_h is +0.28% and 0.02%, respectively.

Intuitively, it is easier for $\{H_t\}$ to hit the lower or upper bound when σ_h increases. $\{H_t\}$ hitting the lower bound will trigger defaults by definition and hitting the upper bound may compel only real estate investors (not regular borrowers) to prepay the loans. Either default or prepayment will decrease future cash payments by borrowers, and thereby reduce the MBS value. However, the probability of $\{H_t\}$ staying within the bounds will also increase when σ_h increases. If $\{H_t\}$ stay within the bounds and $\{r_t\}$ never hit the lower bound at time t as indicated in Table 1, all of the borrowers may choose to pay the monthly payments regularly until default or prepayment occurs. If so, the number of future cash payments by borrowers will increase, which will increase the MBS value thereafter. Hence, the MBS value may increase as a result of the effects of the "longevity" of the mortgages which outweigh the effect of default or prepayment as σ_h increases.

Figure 5 Relationship between Interest Rate Volatility and Non-Agency MBS Value

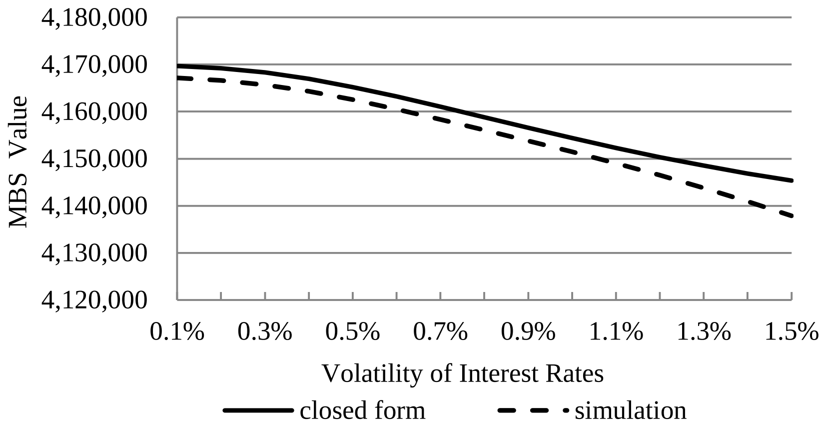
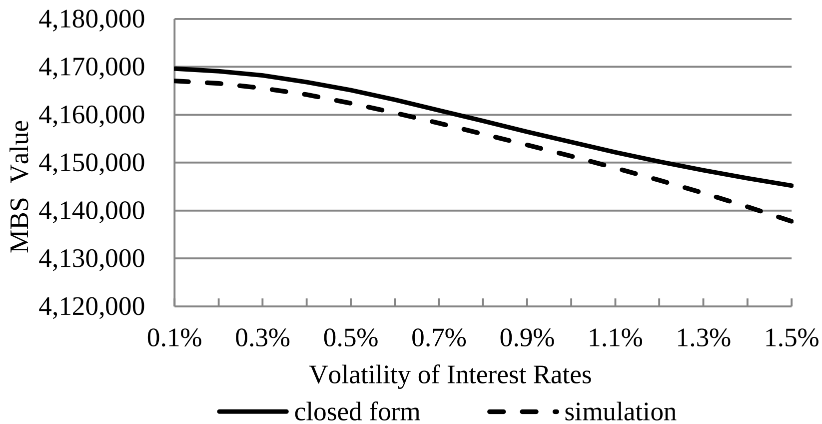


Figure 6 Relationship between Interest Rate Volatility and Agency MBS Value



Interestingly, Figures 7 and 8 show that the MBS value is positively related to σ_h, but the marginal effects of σ_h on non-agency and agency MBS values are quite different. As σ_h increases from 6% to 7%, the percentage change of the non-agency MBS values increases from +0.08% to +0.22% while that of agency MBS values remains the same at +0.04%. Figure 7 shows that the non-agency MBS value increases with σ_h more rapidly when $\sigma_h \geqq 7\%$. The average percentage price change of non-agency MBSs per 1% change of σ_h is +0.58% when $\sigma_h \in [7\%, 10\%]$ while +0.03% if $\sigma_h \in [1\%, 7\%)$. In Figure 8, we see that the agency MBS value is positively related to σ_h y until it reaches 7%, and the slope decreases afterwards.

Table 8 Relationship between Housing Price Volatility (σ_h) and MBS Value

	Non-Agency MBS			Agency MBS		
σ_h	closed form	simulation	error	closed form	simulation	error
1%	4150612.36	4147815.23	0.07%	4150612.36	4147815.23	0.07%
2%	4151015.77	4148204.05	0.07%	4151015.77	4148204.05	0.07%
3%	4151743.15	4148904.86	0.07%	4151743.15	4148904.86	0.07%
4%	4152832.68	4149954.78	0.07%	4152831.94	4149954.04	0.07%
5%	4154389.13	4151459.11	0.07%	4154277.41	4151347.59	0.07%
6%	4157696.10	4154702.77	0.07%	4155969.74	4152979.22	0.07%
7%	4166680.41	4163615.19	0.07%	4157608.15	4154556.25	0.07%
8%	4185603.51	4182464.87	0.08%	4158811.19	4155706.68	0.07%
9%	4215782.36	4212580.48	0.08%	4159321.77	4156179.36	0.08%
10%	4255302.63	4252060.01	0.08%	4159073.29	4155908.94	0.08%

Figure 7 Relationship between Housing Price Volatility and Non-Agency MBS Value

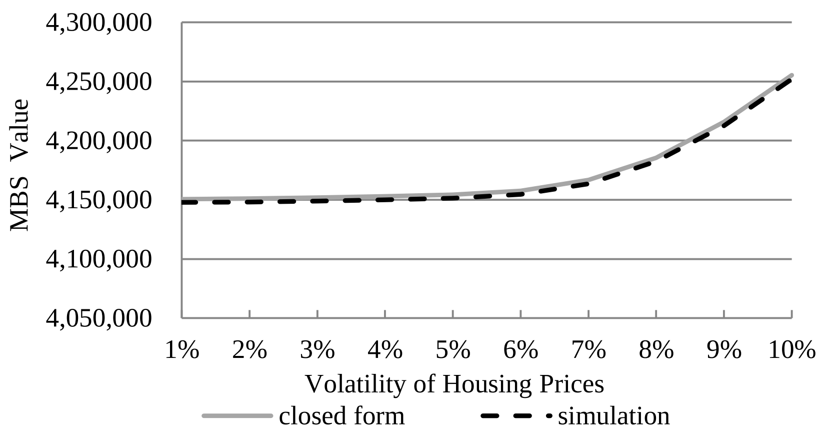
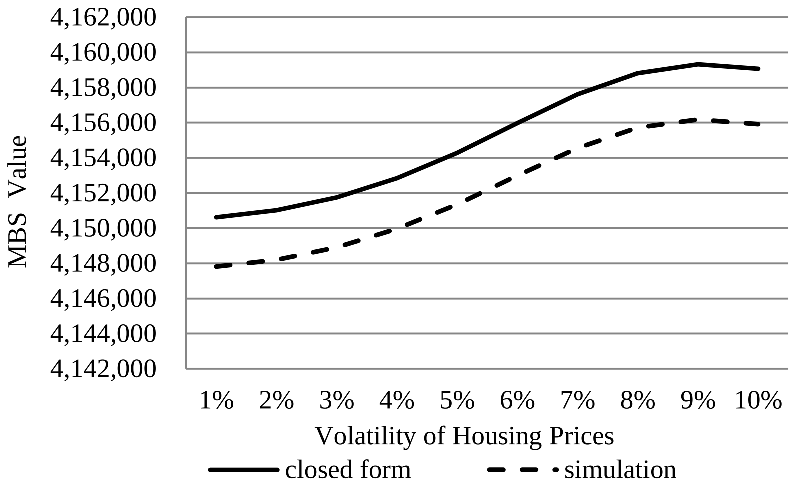


Figure 8 Relationship between Housing Price Volatility and Agency MBS Value



5. Conclusion

Pricing MBSs with an embedded barrier option feature is the main purpose of our study. We focus on the prepayment behavior of borrowers while the type of borrower and his/her optimal prepayment choice is endogenously determined. Using the FHT as a proxy for the trigger point of prepayment is our logical approach to solving the valuation problem. The unique perspective of our study is that we hypothesize that "real estate investors" among all borrowers behave differently from "regular borrowers"; that is, real estate investors are more likely to prepay their loans. Our contribution is threefold: (1) to the best of our knowledge, our paper is one of few studies to apply the concept of barrier option valuation to price MBSs and provide a model by using FHT to arrive at a true value rather than the confidence interval obtained by simulation; (2) we provide evidence to show that the proportion of real estate investors to all borrowers is a determining factor while valuing MBSs, and that the concentration of investor borrowers affects agency and non-agency MBS values differently; and (3) we examine and quantify the marginal effects on the MBS value of four factors that affect either the amount of cash payment from borrowers or the trigger timing of prepayment.

We find that the MBS value is positively related to loan to value (*LTV*) and house price volatility (σ_h) while negatively related to the proportion of real estate investors (ρ) and interest rate volatility (σ_r). The average impact per 1% change of (*LTV*, ρ, σ_r) on both non-agency and agency MBS values obtained by our pricing formula is (+3.34%, -0.42%, -0.42%), respectively. However, the percentage change of the non-agency and agency MBS values per 1% change of σ_h is +0.28% and 0.02%, respectively. We also find evidence which shows that MBS value may increase as a result of the effects of the "longevity" of mortgages, which outweigh the effect of default or prepayment as house price volatility increases. This result is interesting as it is opposite to what we had expected. For robustness testing, we compare the pricing difference between our closed-form solution and Monte Carlo simulation in valuing MBSs. We find that the MBS value from our formula is slightly higher than that of the Monte Carlo simulation, and the difference is less than 0.2%.

Furthermore, it is worth noting that the plotted patterns of the marginal effect of each factor on the MBS value is the same for non-agency and agency MBSs, except for house price volatility. The non-agency MBS value increases with house price volatility more rapidly when housing price volatility is ≧7%. However, the agency MBS value is positively related to house price volatility until it reaches 7%, and the slope decreases afterwards. It is an interesting empirical question to see if the different plotted patterns in Figures 7 and 8 are due to the involvement of real estate investors as borrowers, other than the fact that most non-agency MBSs are backed by subprime mortgages while agency MBSs by prime loans. The optimal prepayment choices of real estate investors as borrowers may be the reason why house price volatility affects non-agency and agency MBS values differently. Our results enrich the possible paths of the MBS price formation process and shed light on further development of the MBS market.

Certainly, Monte Carlo simulation can be used in this study to calculate the price. However, the simulation results may cause sampling errors. The result is just the true price of the point estimation or the confidence interval and this will cause the incorrect estimation of the hedging parameters. Therefore, we feel that it takes virtually the same amount of computation time to obtain the results by simulation or using a closed form solution. However, our closed form model is a faster pricing tool than Monte Carlo simulation while retaining more accuracy and consistency than the hazard model approach.

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Appendices

Appendix I Cumulative Density Functions G_L and G_U

$$
\begin{aligned}
G_L(t;H_0,L,U) &= \int_0^t g_L(s;H_0,L,U)ds \\
&= \left[\frac{L}{H_0}\right]^{\left(\mu_h \sigma_h^{-2}-\frac{1}{2}\right)} \{\sum_{n=0}^{\infty}[e^{-a_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(\frac{\left(\mu_h-\frac{\sigma_h^2}{2}\right)t-\sigma_h a_h}{\sigma_h\sqrt{t}}) \\
&\quad + e^{a_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h a_h}{\sigma_h\sqrt{t}})] \\
&\quad - \sum_{n=-\infty}^{-1}[e^{a_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h a_h}{\sigma_h\sqrt{t}}) \\
&\quad + e^{-a_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t-\sigma_h a_h}{\sigma_h\sqrt{t}})]\}
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
G_U(t;H_0,L,U) &= \int_0^t g_U(s;H_0,L,U)ds \\
&= \left[\frac{U}{H_0}\right]^{\left(\mu_h \sigma_h^{-2}-\frac{1}{2}\right)} \{\sum_{n=0}^{\infty}[e^{-b_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(\frac{\left(\mu_h-\frac{\sigma_h^2}{2}\right)t-\sigma_h b_h}{\sigma_h\sqrt{t}}) \\
&\quad + e^{b_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h b_h}{\sigma_h\sqrt{t}})] \\
&\quad - \sum_{n=-\infty}^{-1}[e^{b_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t+\sigma_h b_h}{\sigma_h\sqrt{t}}) \\
&\quad + e^{-b_n\left(\mu_h \sigma_h^{-1}-\frac{\sigma_h}{2}\right)}\Phi(-\frac{\left(\mu_h-\frac{\sigma_h^{2}}{2}\right)t-\sigma_h b_h}{\sigma_h\sqrt{t}})]\}
\end{aligned}
\tag{A2}
$$

Appendix II

Since we know that $\{r_t\}$ is stochastic, we use a series of discount rates to discount the cash payments at time t described in Section 3.3 and obtain the corresponding present value in the event of prepayment (PV_{1t}), default (PV_{2t}^{a} and PV_{2t}^{b}), and holding-on until loan maturity (PV_{3T}). Recall that PV_{2t}^{a} is the present value of cash payments for non-agency MBSs and PV_{2t}^{b} for agency MBSs. We derive the expected values of ($PV_{1t}, PV_{2t}^{a}, PV_{2t}^{b}, PV_{3T}$) below, denoted as ($V_{1t}, V_{2t}^{a}, V_{2t}^{b}, V_{3T}$), respectively.

If $X \sim N(\mu,\sigma^2) \Rightarrow E(e^{tX}) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$, we let $Y = e^X$. Then

$$\begin{cases} X \sim \ln N(\mu,\sigma^2) \\ E(Y) = E(e^X) = \exp(\mu + \frac{1}{2}\sigma^2) \end{cases}$$

From Equation (15),

$$\begin{aligned} R(t) &= \int_0^t r_s ds \\ &\sim N\left(\mu_r t + \left(r_0 - \mu_r\right)\frac{1-e^{-\theta t}}{\theta}, \frac{\sigma_r^2}{\theta^2}\left(t - \frac{1-e^{-\theta t}}{\theta} - \frac{1}{2}\theta\left(\frac{1-e^{-\theta t}}{\theta}\right)^2\right)\right) \end{aligned}$$

We define

$$M_{R,t} = \mu_r t + (r_0 - \mu_r)\frac{1-e^{-\theta t}}{\theta}$$

$$S_{R,t} = \frac{\sigma_r^2}{\theta^2}\left(t - \frac{1-e^{-\theta t}}{\theta} - \frac{1}{2}\theta\left(\frac{1-e^{-\theta t}}{\theta}\right)^2\right)$$

then

$$E\left(\exp(-\int_0^t r_s ds)\right) = exp\left(-M_{R,t} + \frac{S_{R,t}}{2}\right)$$

$$
\begin{aligned}
V_{1t} &= E\left[PV_{1t}\right] \\
&= E[PMT \times (e^{-\int_0^{\frac{1}{12}} r_s ds} + e^{-\int_0^{\frac{2}{12}} r_s ds} + \ldots + e^{-\int_0^{t} r_s ds}) + UPB_t \times e^{-\int_0^{t} r_s ds} \\
&= PMT \times \sum_{i=1/12}^{t} E(e^{-\int_0^{i} r_s ds}) + UPB_t \times E(e^{-\int_0^{t} r_s ds}) \\
&= PMT \times \sum_{i=1/12}^{t} \exp(-M_{R,i} + \frac{S_{R,i}}{2}) \\
&\quad + UPB_t \times \exp(-M_{R,t} + \frac{S_{R,t}}{2})
\end{aligned}
$$

$$
\begin{aligned}
V_{2t}^{a} &= \mathrm{E}[PV_{2t}^{a}] \\
&= E[PMT \times (e^{-\int_0^{\frac{1}{12}} r_s\, ds} + e^{-\int_0^{\frac{2}{12}} r_s\, ds} + \ldots\ldots + e^{-\int_0^{t-\frac{1}{12}} r_s\, ds}) + H_t \times e^{-\int_0^{t} r_s\, ds}] \\
&= PMT \times \sum_{i=\frac{1}{12}}^{t-\frac{1}{12}} \exp\left(-M_{R,i} + \frac{S_{R,i}}{2}\right) + H_0
\end{aligned}
$$

$$
\begin{aligned}
V_{2t}^{b} &= \mathrm{E}\left[PV_{2t}^{b}\right] \\
&= E[PMT \times (e^{-\int_0^{\frac{1}{12}} r_s\, ds} + e^{-\int_0^{\frac{2}{12}} r_s\, ds} + \ldots\ldots + e^{-\int_0^{t-\frac{1}{12}} r_s\, ds}) + UPB_t \times e^{-\int_0^{t} r_s\, ds}] \\
&= PMT \times \sum_{i=\frac{1}{12}}^{t-\frac{1}{12}} \exp\left(-M_{R,i} + \frac{S_{R,i}}{2}\right) + UPB_t \times \exp\left(-M_{R,t} + \frac{S_{R,t}}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
V_{3T} &= E\left[PV_{3T}\right] \\
&= E[PMT \times (e^{-\int_0^{\frac{1}{12}} r_s\, ds} + e^{-\int_0^{\frac{2}{12}} r_s\, ds} + \ldots\ldots + e^{-\int_0^{T} r_s\, ds})]. \\
&= PMT \times \sum_{i=\frac{1}{12}}^{T} \exp\left(-M_{R,i} + \frac{S_{R,i}}{2}\right)
\end{aligned}
$$

QED