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Private Real Estate Investment Analysis within a One-Shot Decision Framework

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Land development is a typical one-shot decision for private investors due to the huge investment expense and the fear of substantial loss. In this paper, a private real estate investment problem is analyzed within a one-shot decision framework, which is used for a situation where a decision is made only once. The one-shot decision framework involves two steps. The first is to identify which state of nature should be focused for each alternative. The second is to evaluate alternatives by using the focused states of nature. In a one-shot decision framework, the behavior of different types of private investors, such as normal, active, passive and more easily satisfied ones, are examined. The analysis provides insights into personal real estate investment and important policy implications in the regulation of urban land development.

Keywords

Private real estate investment; Possibility theory; One-shot decision; Focus points

1. Introduction

There are many underutilized and vacant urban lots throughout the world, which are held by private investors who are interested in maximizing their wealth by land development. Three approaches are commonly used for property and land valuation (Appraisal Institute, 2001; Baum and Crosby, 1988; Isaac, 2002). The first is the cost approach, which estimates the property by summing the land value and the depreciated value of any improvements. The second is the sales comparison approach which compares the characteristics of a subject property with those of comparable properties sold in similar transactions. This kind of method is a variation of hedonic regression models. The third is income approach. The discounted cash flow (DCF) model is one of income approaches where appraisers determine the most probable use of the land, appraise the property according to the use, and then discount the future value, such as rental rate for condominiums and capital gain in the real estate market, to the present. The above three approaches are used for the valuation of land as a site for the construction of a particular building at the current time. Which approach is more applicable is problem-specific. For example, in a situation where the purchase of real estate is for investment, the income approach seems more suitable, whereas for individual use, the sales comparison approach is more acceptable. Another way for the valuation of land is to consider a vacant piece of land as a potential site for development in the future so that it is viewed as an option for purchasing one of a number of different possible buildings at exercise prices that are equal to their respective construction costs. Real option based valuation methods are proposed in the literature (Titman, 1985; Williams, 1991; Amram and Kulatilaka, 1999). Seiler et al. (2008) examines residential real estate from the viewpoint of behavioral finance theories. The real estate decision model is built based on the regret theory with false reference points. Regret aversion and false reference points are statistically tested in a hypothetical real estate investment with an induced false reference point. Some research studies focus on testing psychological effects for property valuation within the framework of the Prospect theory (Diaz and Hansz, 1997; Gallimore, 1996). Mori et al. (2009) tests the reflection hypothesis in the Prospect theory as a contributing factor in a borrower's choice of mortgage type.

Housing price is a key factor for real estate investment decision making. However, it is always difficult to obtain the pricing function. Kummerow (2002) points out the following characteristics of real estate assets. Real estate assets are heterogeneous. There are hundreds of factors, such as age, location, site/view, design/appeal, room count/gross living area, functional utility, etc., which might affect prices in various situations. Moreover, properties trade infrequently, perhaps once every 5-10 years for the average house. The amount of sales evidence varies widely in particular cases, but generally, there are few sales of properties which are similar enough to be "comparable" and none of identical properties. The inherent uncertainty in property prices are traditionally characterized by probability distributions (Ratcliff, 1972 and Squirell, 1985). However, the lack of precise information on properties often poses a problem. As mentioned by Bagnoli and Smith (1998), appraisers use a great deal of judgment to identify the characteristics (attributes) of properties that relate to property prices. Additionally, they usually have to consider qualitative characteristics, such as structural quality, architectural attractiveness and location convenience. In order to express the inherent imprecision in the ways that people think and make decisions about the pricing of real estate, they propose a fuzzy logic based model to obtain the possibility distribution (fuzzy membership function) of the property price and show its effectiveness. Combining fuzzy sets and artificial neural network, Guan et al. (2008) proposes an adaptive neuro-fuzzy inference system based approach to real estate property assessment and the real estate market in the midwest region of the United States is analyzed.

Private real estate investment is a typical one-shot decision problem for personal investors due to the huge investment expense and the fear of substantial loss. There is only one outcome for a one-shot decision problem. The private real estate investment problem is analyzed within a one-shot decision framework. The procedure for a one-shot decision consists of two steps. In the first step, for each alternative, some states of nature, which are called focus points, are selected amongst all states of nature based on the attitudes of decision makers on uncertainty. In the second step, alternatives are evaluated based on their focus points to obtain the optimal alternative. For a private real estate investment problem, the state of nature is a building price in the future. The building price is characterized by the possibility distribution to reflect the degree to which the price will occur in the future. The alternative is a building size that is determined by a personal real estate investor. Within a one-shot decision framework, the housing price is selected by each building size as its focus point in the first step. In the second step, the optimal building size is obtained to maximize the satisfaction when the focus points come true. The behavior of different types of investors, such as normal, active, passive and more easily satisfied ones are examined.

The purpose of this study is to examine a current land development decision within a one-shot decision framework. The main contributions of this study are as follows. A private real estate investment decision model is proposed within a one-shot decision framework, which is an innovative study on analyzing real estate investment problems from an economic viewpoint by using tools from the possibility theory. The analysis in this paper demonstrates the relation between the amount of uncertainty and the investment scale for different types of personal investors. The proposed model provides insights into personal real estate investment decisions and important policy implications in the regulation of urban land development.

This paper is organized as follows: Section 2 provides some basic ideas of possibility. In Section 3, the one-shot decision framework is addressed. In Section 4, a possibilistic decision model for private real estate investment is proposed. Finally, concluding remarks for this research are made.

2. Basic Idea of Possibility

Possibilities can be explained from several semantic aspects. One way to explain possibility is ease of achievement. Another way is plausibility; that is, the propensity of an event to occur, which relates to the concept of "potential surprise". A third way to explain possibility is logical consistency with available information. Finally, possibility can be explained as preference, referring to the willingness of an agent to make a decision. Possibility is dually related to necessity in the sense that "not A" being not possible means that A is necessary. A semantic analysis of necessity can be done through reference to credibility, acceptance and fact. Possibility theory is based on two non- additivity measures; possibility and necessity. A detailed discussion about the differences between possibility theory and probability theory can be found in the literature (Dubois, 2006) and the very basics of possibility theory is introduced in Appendix A. Possibility distribution is a function whose value shows the degree to which an element is to occur, defined as follows:

Definition 1. Given a function $\pi : S \to [0,1]$, if $\max_{\pi \in S} \pi(x) = 1$, then the function $\pi(x)$ is called a possibility distribution where S is the sample space with an element x. $\pi(x)$ is the possibility degree of x. $\pi(x) = 1$ means that it is normal that x occurs and $\pi(x) = 0$ means that it is abnormal that x occurs. A smaller possibility degree of x means more surprise for the occurrence of x. The following two examples are used to illustrate possibility.

Example 1 How Many Eggs Will Peter Eat This Morning?

We can predict the amount of eggs eaten by Peter from two aspects, probability (frequency) and possibility (capability). Assume that the sample space S is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The estimation is listed in Table 1. It follows from Table 1 that Peter often eats one egg in the morning, sometimes two or three eggs from a statistical viewpoint, but eating one, two or three eggs is normal without any unpleasantness for him. Even if he never eats four, five, six or seven eggs in the morning, he can do it if he tries. It is impossible that he eats eight or nine eggs in the morning.

Example 2 Who Is A Criminal?

A car has been destroyed by somebody in a parking lot. After careful investigation, it is sure that one and only one of three suspects; A, B or C, must be a criminal, but the exact criminal is still unknown. Suppose that based on the currently obtained evidence, subjective probabilities are used to characterize the belief about the criminal amongst these three suspects and given as follows: $P(A) = 0.4$, $P(B) = 0.4$ and $P(C) = 0.2$. In consideration of the relation $P(A) = 1 - P(\overline{A})$ where $\overline{A}$ is the complement of A, it can be concluded that there is no criminal amongst the three suspects in the context of probability ($P(A) < P(\overline{A}), P(B) < P(\overline{B}), P(C) < P(\overline{C})$). This conclusion is in conflict with the antecedent one; that is, one and only one of the three suspects; A, B or C, must be a criminal. This conflict originates from the existence of incomplete information. In this example, the possibility distributions that show the degrees to which a person might be a criminal are given as follows: $\pi(A) = 1$, $\pi(B) = 1$and $\pi(C) = 0.7$. $\pi(A) = \pi(B) = 1$ means that based on the obtained evidence, A and B have the highest possibility of committing a crime. The relation $\pi(A) \neq 1 - \pi(\overline{A})$ implies that the possibility degree of A being a criminal does not provide any information on A not being a criminal.

It follows from this example that possibility distribution is a less restricted framework than single probability measures and hence, can be used for encoding ill-known subjective probability information. From the above examples, it is clear that the possibility distribution can be used to represent the knowledge or judgment of human beings. Bagnoli and Smith (1998) employ the fuzzy logic based model to obtain the possibility distribution (fuzzy membership function) of the property price. Guo and Tanaka (2003) propose the methods for identifying the upper and lower possibility distributions from the given possibility degrees of samples via linear programming problems. The identified possibility distributions of the returns of securities are used for portfolio selection problems.

Table 1 Possibilities Versus Objective Probabilities

Eggs	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Probabilities	0.6	0.3	0.1	0	0	0	0	0	0
Possibilities	1	1	1	0.8	0.4	0.2	0.1	0	0

3. One-Shot Decision Framework

The first step in a decision analysis is the problem formulation. The set of an alternative *a* is *A*. The set of a state of nature *x* is *S*. The degree to which a state of nature is to occur in the future is characterized by a possibility distribution $\pi(x)$ defined in Definition 1. The consequence that results from the combination of an alternative *a* and a state of nature *x* is referred to as a payoff, denoted as $\upsilon(x,a)$. The satisfaction level of a decision maker for a payoff can be expressed by a satisfaction function, as defined below.

Definition 2. The set of a payoff υ (x,a) is denoted as V. The following function,

$$u : V \to [0,1] \tag{1}$$

$$\text{with } u(v_1) > u(v_2) \text{, for } v_1 > v_2 , \tag{2}$$

is called a satisfaction function. As the payoff is the function of x and a, we can rewrite the satisfaction function as $u(v(x,a))$. For the sake of simplification, sometimes we write $u(v(x,a))$ as $u(x,a)$ in this paper.

In a decision analysis, a decision maker has to select an alternative before the state of nature is known. In some cases, the selected alternative may provide a good or excellent result due to a favorable state of nature. In other cases, a relatively unfavorable state of nature may occur which causes the selected alternative to provide only a fair or even poor result. It is true that one and only one state of nature will occur for a one-shot decision problem. The information for a one-shot decision can be summarized as a quadruple (A, S, π, u). A one- shot decision is to choose one alternative based on (A, S, π, u) when only one decision chance is given. The procedure for making a one-shot decision is divided into the following two steps (Guo, 2004, 2009).

Step 1 Decide which state of nature should be considered for each alternative. The following three choices are used for selecting some states of nature. As three choices characterize the different attitudes of decision makers on uncertainty, a decision maker should decide which choice will be employed to fit his own type in this step.

Choice 1 A decision maker only focuses on the normal case so that the state of nature with possibility degree 1, denoted as x^o, is chosen, which is:

$$x^o = \arg\max_{x \in S} \pi(x) \cdot \tag{3}$$

x^o is called a normal focus point. $\arg\max_{x \in S} f(x)$ is used to denote the value of x which makes $f(x)$ maximize.

Choice 2 For an alternative a, a decision maker chooses a state of nature, denoted as $x^*(a)$, which is:

$$x^*(a) = \arg\max_{x \in S} \min(\pi(x), u(x,a)) \cdot \tag{4}$$

It follows from (4) that $x = x^*(a)$ makes $h(x,a) = \min(\pi(x), u(x,a))$ reach the maximum. $\min(\pi(x), u(x,a))$ represents the lower bound of the pair $(\pi(x), u(x,a))$. It means that the state of nature x's possibility degree and satisfaction level provided by an alternative a are at least $\min(\pi(x), u(x,a))$.

Increasing $\min(\pi(x), u(x,a))$ ($\max_{x \in S} \min(\pi(x), u(x,a))$) will increase the lower bound of the possibility degree and satisfaction level simultaneously. Therefore, $\arg\max_{x \in S} \min(\pi(x), u(x,a))$ is for seeking the state of nature with the higher possibility degree and satisfaction level. The selected state of nature $x^*(a)$ is called an active focus point of the alternative *a*.

Choice 3 For an alternative *a*, a decision maker chooses a state of nature, denoted as $x_*(a)$, which is:

$$x_*(a) = \arg\min_{x \in S} \max(1 - \pi(x), u(x,a)) \cdot \tag{5}$$

It follows from (5) that $x = x_*(a)$ makes $g(x,a) = \max(1 - \pi(x), u(x,a))$ reach the minimum. $\max(1 - \pi(x), u(x,a))$ represents the upper bound of the pair $(1 - \pi(x), u(x,a))$. It means that the state of nature x 's possibility degree is at least $1 - \max(1 - \pi(x), u(x,a))$ and its satisfaction level provided by an alternative a is at most, $\max(1 - \pi(x), u(x,a))$. Decreasing $\max(1 - \pi(x), u(x,a))$ ($\min_{x \in S} \max(1 - \pi(x), u(x,a))$) will increase the lower bound of the possibility degree $1 - \max(1 - \pi(x), u(x,a))$ and decrease the upper bound of the satisfaction level $\max(1 - \pi(x), u(x,a))$. Therefore, $\arg\min_{x \in S} \max(1 - \pi(x), u(x,a))$ is for seeking the state of nature with a higher possibility degree and lower satisfaction level. The selected state of nature $x_*(a)$ is called a passive focus point of the alternative *a*.

Example 3 Let us consider a discrete case for private real estate investment. The set of states of nature $S = \{x_1, x_2, x_3, x_4, x_5\}$ consists of five housing prices per building size in the future and the set of alternatives $A = \{a_1, a_2\}$ includes two housing sizes. The possibility degrees of housing prices and satisfaction levels for the two housing sizes in each housing price are listed in Table 2. The private investor has only one chance to choose one building size from A.

Since $\max_{x \in S} \pi(x) = \pi(x_4) = 1$ holds, we have:

$$x^o = \arg\max_{x \in S} \pi(x) = x_4, \tag{6}$$

which is a normal focus point (building price 4). As x_2 and x_4 make $\min(\pi(x), u(x,a_1))$ ($x \in S = \{x_1, x_2, x_3, x_4, x_5\}$) maximize, which is 0.7, we have:

$$x^*(a_1) = \arg\max_{x \in S} \min(\pi(x), u(x,a_1)) = x_2 = x_4, \tag{7}$$

which are active focus points (building prices 2 and 4) of the alternative a_1

(building size 1). As x_3 makes $\max\left(1-\pi(x), u(x, a_1)\right)$ ($x \in S = \{x_1, x_2, x_3, x_4, x_5\}$) minimize, which is 0.5, we have:

$$x_*(a_1) = \arg\min_{x\in S} \max(1-\pi(x), u(x, a_1)) = x_3, \tag{8}$$

which is the passive focus point (building price 3) of a_1 (building size 1). Let us take a close look at Table 2 to understand the above results. The normal focus point is x_4 (building price 4) because its possibility degree is 1. The active focus points of x_1 (building size 1) are x_2 and x_4 (building prices 2 and 4) which implies that (0.7, 1) and (1, 0.7) are undominated by the pairs of the possibility degrees and the satisfaction levels on the other states of nature (building prices 1, 3, 5). Both of them are the states of nature with high possibility degrees and satisfaction levels. The passive focus point of a_1 (building size 1) is x_3 (building price 3), which is a state of nature with a high possibility degree and lower satisfaction level. Likewise, the active focus point of a_2 (building size 2) is x_2 (building price 2) whereas its passive focus point is x_4 (building price 4). It can be seen that different alternatives (building sizes) associate with different focus points (building prices).

Table 2 Information on Possibility Degrees and Satisfaction Levels

	x_1	x_2	x_3	x_4	x_5
$\pi(x_i)$	0.3	0.7	0.75	1	0.6
$1-\pi(x_i)$	0.7	0.3	0.25	0.2	0.4
$u(x_i, a_1)$	0.9	1	0.5	0.7	0.9
$u(x_i, a_2)$	0.6	0.7	0.6	0.55	1

Step 2. Based on the different types of focus points, the optimal alternatives are determined as follows:

$$a^o = \arg\max_{a\in A} \min_{x^o\in X^o} u(x^o, a), \tag{9}$$

$$a^* = \arg\max_{a\in A} \max_{x^*(a)\in X^*(a)} u(x^*(a), a), \tag{10}$$

$$a_* = \arg\max_{a\in A} \min_{x_*(a)\in X_*(a)} u(x_*(a), a), \tag{11}$$

where X^o is the set of normal focus points, $X^*(a)$ and $X_*(a)$ are the sets of active and passive focus points of an alternative a, respectively. a^o, a^* and a_* are called normal, active and passive optimal alternatives, respectively. $x^*(a^*)$ and $x_*(a_*)$ are called optimal active and optimal passive focus points, respectively.

It follows from (9), (10) and (11) that when determining which alternative is the best, only the satisfaction levels on focus points are taken into account. The reason is as follows: the decision maker thinks the focus points are the most appropriate states of nature for him/her and the decision maker chooses one alternative which can bring about the best consequence once the focus point comes true. Decision rules maximin and maximax are used in (9), (10) and (11) for the cases where multiple focus points exist for an alternative. Maximin and maximax reflect the conservative and aggressive attitudes, respectively.

Let us go back to Example 3. We have $X^o = \{x_4\}$, $X^*(a_1) = \{x_2, x_4\}$, $X_*(a_1) = \{x_3\}$, $X^*(a_2) = \{x_2\}$ and $X_*(a_2) = \{x_4\}$. When $a = a_1$, $\min_{x^o \in X^o} u(x^o, a)$ ($a \in \{a_1, a_2\}$) reach maximum 0.7. According to (9), we have $a^o = a_1$. $a = a_1$ makes $\max_{x^*(a) \in X^*(a)} u(x^*(a), a)$ ($a \in \{a_1, a_2\}$) reach maximum 1 and $a = a_2$ makes $\min_{x_*(a) \in X_*(a)} u(x_*(a), a)$ ($a \in \{a_1, a_2\}$) reach maximum 0.55. According to (10) and (11), we have $a^* = a_1$ and $a_* = a_2$. The optimal active focus point $x^*(a^*)$ and the optimal passive focus point $x_*(a_*)$ are $x^*(a_1) = x_2$ and $x_*(a_2) = x_4$, respectively. Since one and only one price in A will occur in the future, the private real estate investor must focus on one of them for making a decision. It is clear that if he makes a decision based on the normal case, he will choose a_1 (building size 1); if he thinks that x_2 (building price 2) will occur, then he will choose a_1 (building size 1); if he thinks that x_4 (building price 4) will occur, then he will choose a_2 (building size 2).

The one-shot decision framework is enlightened by a common phenomenon: when you ask a person why s/he makes such a one-shot decision with little information, s/he will always tell you just one scenario which is crucial to him/her and is the basis for the obtained decision. Taking into account the crucial scenario for making a decision corresponds to choosing focus points amongst all of the states of nature with regards to plausibility and satisfaction. The proposed framework helps a decision maker in finding out the best solution in accordance to his/her attitude on plausibility and satisfaction. Alternatives choose their own favorable states of nature (focus points) in Step 1 and compare consequences on the focus points with each other to obtain the optimal alternative in Step 2. This kind of consideration is similar to the idea of the data envelopment analysis (DEA), in which each decision making unit (DMU) selects its own favorite weights of inputs and outputs for calculating its own efficiency (Charnes et al., 1978; Guo and Tanaka, 2001; Guo et al., 2000). The comparison of Choices 2 and 3 with optimistic and pessimistic utilities, proposed by Yager (1979) and Whalen (1984), respectively, is given in Appendix B.

4. Possibilistic Models for Private Real Estate Investment

Land development is a typical one-shot decision for private investors due to the huge investment expense and the fear of substantial loss. A house in this research is characterized by its size or the number of units, q, for simplification. The cost of constructing a house on a given piece of land, C, is a strictly increasing and convex differentiable function of the number of units q; that is,

$$dC / dq > 0, \tag{12}$$

$$d^2 C / d^2 q > 0. \tag{13}$$

The rationale for the second assumption is that as the number of floors in a building increases, the foundation of the building must be stronger (Titman, 1985). The profit r that a landowner can obtain by constructing a q-size building is as follows:

$$r(p,q) = pq - C(q), \tag{14}$$

where p is the market price per building size at the end of construction. The building size that maximizes the profit of an investor will satisfy the following maximization problem:

$$R(p) = \max_q r(p,q) = \max_q pq - C(q). \tag{15}$$

In differentiating (15) with respect to q, it follows that the solution to this maximization problem is a building size, which satisfies the following condition in consideration of assumption (13):

$$dC(q) / dq = p. \tag{16}$$

In denoting the solution of (16) as $q^{\nabla}(p)$, the maximal profit of a landowner is as follows:

$$R(p) = \max_q r(p,q) = pq^{\nabla}(p) - C(q^{\nabla}(p)). \tag{17}$$

Theorem 1 $R(p)$ is a strictly increasing and convex continuous function of p.

Proof. See Appendix C.

The fact that the maximal profit of an investor is a strictly increasing and convex continuous function of housing price implies that if a rational decision is made for land development, then not only the profit, but also the marginal profit will increase with the building price increasing.

Corollary 1 The optimal building size $q^{\nabla}(p)$ is a strictly increasing function of the building price p; that is, $\frac{dq^{\nabla}(p)}{dp} > 0$.

Assume that the possibility distribution of the price p, denoted as π_P, is given by the following unimodal continuous function:

$$\pi_P : [p_l, p_u] \to [0,1], \tag{18}$$

where $\pi_P(p_l) = 0, \pi_P(p_u) = 0$ and $\exists p_c \in [p_l, p_u]$ such that $\pi_P(p_c) = 1$. π_P increases within $[p_l, p_c]$ and decreases within $[p_c, p_u]$. p_l and p_u are the lower and upper bounds of prices, respectively, and p_c is the most possible price.

It follows from Corollary 1 that $q^{\nabla}(p_u)$ should be the largest size of the building. As $r(p,q)$ is a strictly increasing function of the building price p, and a concave function of building size q, the lower bound of profit will be $r(p_l, 0) \wedge r(p_l, q^{\nabla}(p_u))$. Without loss of generality, suppose that the relation $r(p_l, q^{\nabla}(p_u)) = p_l q^{\nabla}(p_u) - C(q^{\nabla}(p_u)) \le r(p_l, 0) = 0$ holds. Thus, the range of profit is $[p_l \cdot q^{\nabla}(p_u) - C(q^{\nabla}(p_u)), R(p_u)]$. Let us now formalize the real estate investment problem by (A, S, π, u). The set of alternatives A is $[0, q^{\nabla}(p_u)]$. The set of the states of nature S is $[p_l, p_u]$. π_P is given by (18). $u(r(p,q))$ is a strictly increasing function of the profit r defined as $u : [p_l \cdot q^{\nabla}(p_u) - C(q^{\nabla}(p_u)), R(p_u)] \to [0,1]$.

Remark 1 $u(r(p,q))$ is a continuous, strictly increasing, function of p.

It follows from (14) that $r(p, q)$ is a continuous, strictly increasing, function of p. As $u(r(p, q))$ is a strictly increasing function of $r(p, q)$, $u(r(p, q))$ is a continuous, strictly increasing function of p.

Remark 2 $\max_q u(r(p,q))$ is a continuous strictly increasing function of p.

As $u(r(p, q))$ is a strictly increasing function of $r(p,q)$, we have $\max_q u(r(p,q)) = u\left(\max_q r(p,q)\right)$. It follows from Theorem 1 that $\max_q r(p,q)$ is a continuous, strictly increasing function of p. Thus, $\max_q u(r(p,q))$ is a continuous strictly increasing function of p.

Remark 3 The normal focus point (price) is p_c and the normal optimal alternative (building size) denoted as q^0 is the solution to (16) with $p = p_c$.

Theorem 2

$$\max_q \max_p \min(\pi_P(p), u(r(p,q))) = \max_p \min(\pi_P(p), \max_q u(r(p,q))) \tag{19}$$

$$\max_q \min_p \max(1-\pi_P(p), u(r(p,q))) = \min_p \max(1-\pi_P(p), \max_q u(r(p,q))) \tag{20}$$

Proof. See Appendix D.

Theorem 2 is to obtain the optimal active and passive focus points, which is given by the following corollary.

Corollary 2 The optimal active focus point (building price) p^* and the optimal passive focus point (building price) p_* are obtained as follows:

$$p_* = \arg \min_{p\in[p_l, p_u]} \max(1-\pi_P(p), u(r(q^\nabla(p), p))) \in [p_l, p_c], \tag{21}$$

$$p^* = \arg \max_{p\in[p_l, p_u]} \min(\pi_P(p), u(r(q^\nabla(p), p))) \in [p_c, p_u], \tag{22}$$

where p_* is the horizontal coordinate of the unique intersection of $1-\pi_P(p)$ and $\max_q u(r(p,q))$ within $[p_l, p_c]$ and p^* is the horizontal coordinate of the unique intersection of $\pi_P(p)$ and $\max_q u(r(p,q))$ within $[p_c, p_u]$.

Definition 3 An investor is called a normal, an active or a passive investor if he takes into account, the normal, optimal active or optimal passive focus points for making one-shot decisions.

We can image three types of investors as follows: when making a one-shot decision, the normal investor focuses on the most possible outcome; the active investor takes into account the scenario which can yield a higher satisfaction with a higher possibility; and the passive investor considers the scenario which can lead to a lower satisfaction with a higher possibility.

It follows from Corollary 2 and (16) that the optimal active focus point (building price) p^* satisfies:

$$u(pq - C(q)) = \pi_P(p), \tag{23}$$

$$dC(q)/dq = p, \tag{24}$$

$$p > p_c, \tag{25}$$

and the optimal passive focus point (building price) p_* satisfies:

$$u(pq - C(q)) = 1 - \pi_P(p), \tag{26}$$

$$dC(q)/dq = p, \tag{27}$$

$$p < p_c. \tag{28}$$

Theorem 3 $q^\nabla(p^*)$ and $q^\nabla(p_*)$, denoted as q^* and q_*, are called the active and the passive optimal building sizes, respectively. The active optimal building size is larger than the normal optimal building size, and the normal optimal building size is larger than the passive optimal building size; that is, $q^* > q^o > q_*$.

Proof. Corollary 2 shows $p^* > p_c > p_*$. The following equations

$$dC(q^*)/dq^* = p^*, \tag{29}$$

$$dC(q^o)/dq^o = p_c, \tag{30}$$

$$dC(q_*)/dq_* = p_*, \tag{31}$$

lead to

$$dC(q^*)/dq^* > dC(q^o)/dq^o > dC(q_*)/dq_* \quad . \tag{32}$$

Since $C(q)$ is a convex function, $dC(q^*)/dq^* > dC(q^o)/dq^o > dC(q_*)/dq_*$ implies $q^* > q^o > q_*$.

As the cost of a building is an increasing function of the building size, it follows from Theorem 3 that the decision based on the active focus point yields the larger investment scale than the one based on the normal focus point; the decision based on the normal focus point leads to a larger investment scale than the one based on the passive focus point. The conclusion derived from Theorem 3 that taking a favorable view of real estate investment may stimulate a stronger desire for investment is quite close to common sense in the real world.

Definition 4 Suppose that there are two possibility distributions, π_A and π_B. If for all x, $\pi_A(x) \geq \pi_B(x)$ holds, then π_B is said more informed than π_A, which is denoted as $\pi_B \rhd \pi_A$.

Theorem 4 The active optimal building sizes based on possibility distributions A and B are denoted as q_A^* and q_B^*, respectively, and the passive optimal building sizes based on possibility distributions A and B are denoted as q_{*A} and q_{*B}, respectively. If the possibility distribution B is more informed than the possibility distribution A; that is, $\pi_B \rhd \pi_A$, then the active optimal building size based on the possibility distribution A is not smaller than the one based on possibility distribution B; that is, $q_A^* \geq q_B^*$, and the passive optimal building size based on the possibility distribution A is not larger than the one based on the possibility distribution B; that is, $q_{*A} \leq q_{*B}$.

Proof. Corollary 2 shows that there is one and only one intersection of π_A and $u(r(q^{\nabla}(p), p))$, and also one and only one intersection of π_B and

$u(r(q^{\nabla}(p),p))$ within $p \in [p_c, p_u]$. The horizontal coordinates of these two intersections are denoted as p_A^* and p_B^*, respectively. Within $[p_c, p_u]$, $u(r(q^{\nabla}(p),p))$ strictly increases from $u(r(q^{\nabla}(p_c),p_c))$, $\pi_A(p)$ and $\pi_B(p)$ decrease from $\pi_A(p_c)$ and $\pi_B(p_c)$, respectively. The following relation

$$\pi_A(x) \geq \pi_B(x), \tag{33}$$

implies that $u(r(q^{\nabla}(p),p))$ will intersect with $\pi_B(p)$ not later than with $\pi_A(p)$, so that $p_A^* \geq p_B^*$ holds. As in the proof of Theorem 3, it is straightforward that $q_A^* \geq q_B^*$. Similarly, it is easy to prove $q_{*A} \leq q_{*B}$ because $p_{*A} \leq p_{*B} < p_c$ holds.

From Theorem 4 it can be understood that the increase of uncertainty about the building price can make an investor enlarge his investment scale if he considers the active focus point and reduce his investment scale if he considers the passive focus point. In other words, an obscure business prospect can stimulate the investment of active investors, but deter passive investors from investing. It would be helpful for regulating urban land development to know if the majority of investors in a real estate market are active or passive investors. Assume that the majority of investors are active ones in a particular real estate market. A clear view for the real estate market future from the concerned authorities would be, for example, good medicine for a real estate bubble, whereas a vague forecast or lack of information on the real situation of the real estate market would be beneficial for encouraging the growth of the real estate market. Likewise, we also can make an analysis for a real estate market where the majority of investors are passive.

Definition 5 Suppose that there are two landowners; 1 and 2, and their satisfaction functions are u_1 and u_2, respectively. If for all r, the relation $u_2(r) \geq u_1(r)$ holds, then landowner 2 is said to be more easily satisfied than landowner 1.

Theorem 5 If landowner 2 is more easily satisfied than landowner 1; that is, $u_2(r) \geq u_1(r)$, then for the same possibility distribution of the building price, the optimal building size determined by landowner 2 is always not larger than the one by landowner 1; that is, $q_2^* \leq q_1^*$ and $q_{*2} \leq q_{*1}$.

Proof. Corollary 2 shows that there is a unique intersection of $\pi_P(p)$ and $u_1(r(q^{\nabla}(p),p))$, and also a unique intersection of $\pi_P(p)$ and $u_2(r(q^{\nabla}(p),p))$ within $p \in [p_c, p_u]$. The horizontal coordinates of these two intersections are denoted as p_1^* and p_2^*, respectively. Within $[p_c, p_u]$,

$u_1(r(q^\nabla(p),p))$ and $u_2(r(q^\nabla(p),p))$ strictly increase from $u_1(r(q^\nabla(p_c),p_c))$ and $u_2(r(q^\nabla(p_c),p_c))$, respectively, $\pi_P(p)$ decreases from $\pi_P(p_c)$. The following relation

$$u_2(r(q^\nabla(p),p)) \geq u_1(r(q^\nabla(p),p)) \tag{34}$$

shows that $\pi_P(p)$ will intersect with $u_2(r(q^\nabla(p),p))$ not later than $u_1(r(q^\nabla(p),p))$, so that $p_2^* \leq p_1^*$ holds. Recalling the proof of Theorem 3, it is straightforward that $q_2^* \leq q_1^*$. Similarly, it is easy to prove $q_{*2} \leq q_{*1}$.

Theorem 5 means that independent of whether the investor is active or passive, a more easily satisfied investor results in a smaller building size purchased. As the cost of building is an increasing function of the building size, it can be concluded that a more easily satisfied investor prefers increasingly smaller investment scale. Whether an investor is more easily satisfied is strongly related to the demographical and geographic factors of the investor. Theorem 5 can be used to piece out the situations of the investment scale if the characteristic of the major investors in a particular real estate market is known.

Remark 4 The model used for real estate investment; that is, formula (14) with conditions (12) and (13), is a simple but general model, which has been widely used in economics (Titman, 1985). It is true that real estate investment is a complex process with many different considerations/factors in play. The complexity and uncertainty can be reflected by the possibility distribution of price. As the possibility distribution used in this paper is a unimodal continuous function, which is the most general function, the conclusions obtained in the paper have generality for characterizing investment situations in the real world.

5. Conclusions

A private real estate investment problem is analyzed within a one-shot decision framework. The analysis results provide insights into the behavior of different kinds of personal investors. The normal investor focuses on the most possible outcome; the active investor takes into account the scenario which can yield higher satisfaction with a higher possibility; the passive investor considers the scenario which can lead to lower satisfaction with a higher possibility. The active investor decides on a building size that is larger than the one by a normal investor, which is larger than the one determined by a passive investor. Increasing the uncertainty of the building price can cause an active investor to increase his investment scale and a passive investor to decrease his investment scale. An investor becomes more easily satisfied if his satisfaction level becomes larger for the same profit. A more easily satisfied investor prefers an increasingly smaller investment scale. Such conclusions have important policy implications. For example, if the majority of investors are active in a particular real estate market, a clear view for the real estate market future from the concerned authorities would be good medicine for a real estate bubble, whereas a vague forecast or lack of information on the real situation of the real estate market would be beneficial for encouraging its growth. The situations of the investment scale can be pieced out based on the characteristic of the major investors in a particular real estate market (more easily satisfied or not). An analysis of a private real estate investment problem with possibilistic information is just the beginning. For future research, other kinds of focus points that reflect the different behavior of investors can be used for analyzing real estate investment problems.

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Appendix A The Basics of Possibility Theory (Dubois and Prade, 1988; Klir and Folger, 1988)

Possibility theory is based on possibility and necessity measures which are defined below**.**

Given a universal set X and its power set $\zeta(X)$, a possibility measure, *Pos*, is a function

$$Pos : \zeta(X) \rightarrow [0,1], \tag{A-1}$$

which satisfies the following axiomatic requirements:

(1) $Pos(\Phi) = 0$,

(2) $Pos(\Omega) = 1$,

(3)For any family $\{A_i \mid A_i \in \zeta(X), i \in I\}$, where I is an arbitrary index set,

$$Pos\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} Pos(A_i) \cdot$$

Given a universal set X and its power set $\zeta(X)$, a necessity measure, *Nec*, is a function

$$Nec : \zeta(X) \rightarrow [0,1], \tag{A-2}$$

which satisfies the following axiomatic requirements:

(1) $Nec(\Phi) = 0$,

(2) $Nec(\Omega) = 1$,

(3) For any family $\{A_i \mid A_i \in \zeta(X), i \in I\}$, where I is an arbitrary index set,

$$Nec\left(\bigcap_{i\in I} A_i\right) = \inf_{i\in I} Nec(A_i) \cdot$$

The dual relation between possibility and necessity measures holds as:

$$Nec(A) = 1 - Pos(A^c). \tag{A-3}$$

Suppose that we know the fact $B \subset X$. Based on this fact, the possibility measure of an event A is as follows:

$$Pos(A) = \begin{cases} 1; A \bigcap B \neq \varnothing \\ 0; A \bigcap B = \varnothing \end{cases}, \tag{A-4}$$

and the necessity measure of an event A is as follows:

$$Nec(A) = \begin{cases} 1; A \supseteq B \\ 0; others \end{cases}. \tag{A-5}$$

Appendix B Comparison with Optimistic and Pessimistic Utilities (Guo, 2008, 2009)

The optimistic and pessimistic utilities are defined as follows:

$$u^*(a) = \max_{x \in S} \min(\pi(x), u(x,a)), \tag{B-1}$$

$$u_*(a) = \min_{x \in S} \max(1-\pi(x), u(x,a)). \tag{B-2}$$

(B-1) proposed by Yager (1979) and (B-2) proposed by Whalen (1984) are axiomatized in the style of Savage by Dubois et al. (2001) and called the optimistic and pessimistic utilities of an alternative *a*, respectively. Giang and Shenoy generalize these two utilities by introducing an order on a class of canonical lotteries (2005).

The differences between the one-shot decisions based on Choices 2 and 3 and the decisions based on optimistic and pessimistic utilities are described below.

Comparison 1

In (B-1) and (B-2), $u(x,a)$ is regarded as a fuzzy event. $u^*(a)$ and $u_*(a)$ are the possibility and necessity measures of $u(x,a)$, respectively, which are used for evaluating an alternative a (Whalen, 1984; Yager, 1979). All explanations are based on a strong commensurability assumption between possibility and preference. However, within a one-shot decision framework, the possibility degree and the satisfaction level run on their own rights. In other words, the possibility degree and the satisfaction level do not compare with each other. $\min(\pi(x), u(x,a))$ denotes the lower bound of the pair $(\pi(x), u(x,a))$, which means that the state of nature x 's possibility degree and satisfaction level provided by an alternative a are at least $\min(\pi(x), u(x,a))$. $\max(1-\pi(x), u(x,a))$ denotes the upper bound of the pair $(1-\pi(x), u(x,a))$, which means that the state of nature *x*'s possibility degree is at least $1-\max(1-\pi(x), u(x,a))$ and its satisfaction level provided by an alternative a is at most, $\max(1-\pi(x), u(x,a))$. $\arg\max_{x \in S} \min(\pi(x), u(x,a))$ and $\arg\min_{x \in S} \max(1-\pi(x), u(x,a))$ are for nothing, but seeking some states of nature for an alternative a in Step 1 of the one-shot decision framework.

Comparison 2

Set
$$h(x,a) = \min(\pi(x), u(x,a)). \tag{B-3}$$

Recalling (4), we have:

$$h(x^*(a), a) = \max_{x \in S} h(x, a) = \max_{x \in S} \min(\pi(x), u(x, a)) = \min(\pi(x^*(a)), u(x^*(a), a)) \,. \tag{B-4}$$

Suppose that a^t is the optimal alternative based on the optimistic utility; that is,

$$a^t = \arg\max_{a \in A} \max_{x \in S} \min(\pi(x), u(x, a)) \cdot \tag{B-5}$$

Considering (B-4), we know:

$$a^t = \arg\max_{a \in A} \max_{x \in S} \min(\pi(x), u(x, a)) = \arg\max_{a \in A} h(x^*(a), a) = \arg\max_{a \in A} \min(\pi(x^*(a)), u(x^*(a), a)) \tag{B-6}$$

Recalling (4), we know:

$$a^* = \arg\max_{a \in A} u(x^*(a), a) \cdot \tag{B-7}$$

In a comparison of (B-6) with (B-7), it is clear that a^t and a^* are different in essence. In other words, $\min(\pi(x^*(a)), u(x^*(a), a))$ is used to evaluate an alternative based on the optimal utility whereas $u(x^*(a), a)$ is used to evaluate an alternative within the one-shot decision framework. It follows from (B-6) and (B-7) that the same optimal solution can only be obtained in special cases where $\min(\pi(x^*(a^*)), u(x^*(a^*), a^*)) = u(x^*(a^*), a^*)$. Likewise, we know that the optimal alternative based on the pessimistic utility is different from the one obtained in essence, in accordance to Choice 3. The following numerical example is used to illustrate the above statement.

Example B-1

Suppose the sets of alternatives and states of nature are $A = \{a_1, a_2\}$ and $S = \{x_1, x_2, x_3, x_4, x_5\}$, respectively. The possibility degrees of states of nature and satisfaction levels for two alternatives in each state of nature are listed in Table B-1. If one decision chance is given, which alternative would you like to choose?

From the data shown in Table B-1, we know that the optimistic utilities of alternatives a_1 and a_2 are 0.7 and 0.7001, respectively so that a_2 is selected as an optimistic optimal alternative based on optimistic utilities, whereas the active optimal alternative is a_1 based on (4) and (10). Taking a look at the data in Table B-1, it is easy to accept that a_1 is preferred to a_2 because a_1 dominates a_2 when the states of nature are x_1, x_2, x_4 and x_5; and has almost the same performance as a_2 when the state of nature is x_3.

Table B-1 Data for Comparing Optimistic Utility with Choice 2

	x_1	x_2	x_3	x_4	x_5
$\pi(x_i)$	0.3	0.7	0.700001	0.8	1
$u(x_i, a_1)$	0.9	1	0.699999	0.7	0.6
$u(x_i, a_2)$	0.3	0.2	0.700001	0.2	0.4

Appendix C Proof of Theorem 1

Proof. $q^{\nabla}(p)$ is the implicit function obtained from $f(p,q) = dC(q)/dq - p = 0$ with $f_q(p,q) = d^2C(q)/d^2q \neq 0$. As $f(p,q)$ is a function of class C^1; that is, a 1-times continuously differentiable function, $q^{\nabla}(p)$ is a function of class C^1. Thus, $C(q^{\nabla}(p))$ is also a function of class C^1. As $R(p) = pq^{\nabla}(p) - C(q^{\nabla}(p))$, we have:

$$\begin{aligned}\frac{dR(p)}{dp} &= q^{\nabla}(p) + p\frac{dq^{\nabla}(p)}{dp} - \frac{dC(q^{\nabla}(p))}{dp} \\ &= q^{\nabla}(p) + p\frac{dq^{\nabla}(p)}{dp} - \frac{dC(q^{\nabla}(p))}{dq^{\nabla}(p)}\frac{dq^{\nabla}(p)}{dp}, \\ &= q^{\nabla}(p) + p\frac{dq^{\nabla}(p)}{dp} - p\frac{dq^{\nabla}(p)}{dp} = q^{\nabla}(p) > 0\end{aligned} \tag{C-1}$$

which means that $R(p)$ is a strictly increasing function. Considering $dC(q^{\nabla})/dq^{\nabla} - p = 0$, and using a derivative of an implicit function, we have:

$$\frac{d^2R(p)}{d^2p} = \frac{dq^{\nabla}(p)}{dp} = -\frac{-1}{\dfrac{d^2C(q^{\nabla})}{d^2q^{\nabla}}} = \frac{1}{\dfrac{d^2C(q^{\nabla})}{d^2q^{\nabla}}} > 0, \tag{C-2}$$

which means that $R(p)$ is a convex function of p.

Appendix D Proof of Theorem 2

Proof. Set $h(p,q) = \min(\pi_P(p), u(r(p,q)))$. Considering the following two equations:

$$\max_q \max_p h(p,q) = \max_p \max_q h(p,q), \tag{D-1}$$

$$\max_p \max_q \min(\pi_P(p), u(r(p,q))) = \max_p \min(\pi_P(p), \max_q u(r(p,q))), \tag{D-2}$$

(19) can be easily proved. In what follows, let us consider (20). $1-\pi_P(p)$ is a continuous decreasing function within $[p_l, p_c]$ and $\max_q u(r(p,q))$ is a continuous strictly increasing function of p. The following inequalities

$$\max_q u(r(p_l,q)) < 1-\pi_P(p_l) = 1, \tag{D-3}$$

$$0 = 1-\pi_P(p_c) < \max_q u(r(p_c,q)), \tag{D-4}$$

show that there is one and only one intersection of $1-\pi_P(p)$ and $\max_q u(r(p,q))$ within $[p_l, p_c]$. The horizontal coordinate of this intersection is denoted as p_*, then:

$$1-\pi_P(p_*) = \max_q u(r(p_*,q)) \cdot \tag{D-5}$$

$1-\pi_P(p)$ is a decreasing function within $[p_l, p_*]$, which means for all $p \in [p_l, p_*]$:

$$\max(1-\pi_P(p), \max_q u(r(p,q))) \geq 1-\pi_P(p_*) \cdot \tag{D-6}$$

$\max_q u(r(p,q))$ is an increasing function within $p \in [p_*, p_u]$, which means for all $p \in [p_*, p_u]$:

$$\max(1-\pi_P(p), \max_q u(r(p,q))) \geq \max_q u(r(p_*,q)) \cdot \tag{D-7}$$

(D-5) makes the following hold:

$$\min_p \max(1-\pi_P(p), \max_q u(r(p,q))) = 1-\pi_P(p_*) = \max_q u(r(p_*,q)) \cdot \tag{D-8}$$

Set $q_\Delta = \arg\max_q u(r(p_*,q))$. It can be understood that p_* is also the horizontal coordinate of the intersection of $1-\pi_P(p)$ and $u(r(p,q_\Delta))$. Similarly, because $u(r(p,q_\Delta))$ is a continuous strictly increasing function of p, p_* satisfies $\min_p \max(1-\pi_P(p), u(r(p,q_\Delta)))$. Thus,

$$\min_p \max(1-\pi_P(p), \max_q u(r(p,q))) = \min_p \max(1-\pi_P(p), u(r(p,q_\Delta))) \cdot \tag{D-9}$$

In consideration of (D-9) and the following fact:

$$\min_x \max(f(x), g(x,a)) \leq \min_x \max(f(x), \max_a g(x,a)), \tag{D-10}$$

we have (20).